When is a contest "additive"? A characterization of strategic equivalence*

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Abstract

We characterize the production technologies that generate contests strategically equivalent to the standard additive Lazear–Rosen tournament. A contest with a general production function is strategically equivalent to the additive benchmark if and only if the marginal rate of technical substitution (MRTS) between effort and skill is multiplicatively separable, a characterization that links strategically equivalent contests to the classical theory of additive utility representations. The equivalence class is large: it contains the multiplicative, Cobb–Douglas, CES, and quasilinear technologies, so that, within the class, additivity is a normalization rather than an economic assumption, and existing results for additive contests, e.g., on optimal prize structures and on the effects of noise on effort, extend to the entire class. Strategic equivalence does not, however, make the technology irrelevant for contest design: the equivalence preserves only ordinal information, whereas optimal prize structures depend on cardinal features of the environment. Strategically equivalent technologies can therefore call for opposite prize structures under the same skill distribution. In particular, when skills are exponentially distributed, the case in which all prize structures perform equally well in the additive model, the monotonicity of the MRTS in skill alone determines whether the winner-takes-all or the loser-gets-nothing contest is optimal.

Keywords: contest, strategic equivalence, production function, MRTS, contest design, prize structure

JEL classification: C72, D74, D86

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1. Introduction

Contests are widely used to allocate rewards, promotions, grants, or market opportunities. In many applications, contest outcomes depend on both contestants' efforts and random factors such as talent, productivity shocks, or luck. In a Lazear–Rosen tournament (Lazear and Rosen, 1981), these determinants jointly generate the contestants' outputs, which in turn determine the allocation of prizes. A central ingredient of the model is therefore the production technology that maps effort and the random component, which we call *skill* throughout, into output.[1]

Most of the literature on the Lazear–Rosen tournament assumes an additive production technology of the form $g(e, \theta) = e + \theta$. This specification underlies many influential contributions and has become a standard benchmark in the analysis of tournaments. The additive model is attractive because of its tractability and because it often permits a relatively simple characterization of equilibrium behavior.

At the same time, there are many applications in which additive separability may not be the most natural specification. Effort and skill may interact multiplicatively, they may be complements or substitutes, or they may be linked through more general nonlinear production technologies. Indeed, a variety of non-additive production functions have been studied in the contest literature. While these models often appear substantially different from the additive benchmark, several papers have shown that particular non-additive specifications can be transformed into equivalent additive formulations. This insight is especially prominent in parts of the literature that derive microfoundations of the Tullock contest (Tullock, 1980).[2]

These observations raise a natural question: Which production technologies are strategically equivalent to the standard additive contest in the sense that the equilibrium efforts can be deduced from each other through monotone transformations? Existing results identify particular examples of such equivalence, but a general characterization is missing. The main contribution of this paper is to answer this question and to provide a general characterization

[1]The random component is also called "noise" in the literature. We retain that term when referring to results phrased in that language, as in Section 5.

[2]See, e.g., Jia (2008) and Fu and Lu (2012). For an innovation-contest microfoundation, see Fullerton and McAfee (1999) and Giebe (2014).

of contests which are strategically equivalent to the additive one.

We obtain two main results. First, the equivalence class turns out to be large: it contains the multiplicative specification as well as the Cobb–Douglas, CES, and quasilinear technologies, i.e., specifications that have so far been analyzed separately or connected only case by case. Our characterization shows that these are the same contest after a change of variables. Within the class, additivity is a normalization rather than an economic assumption, and the substantive technological assumption is the multiplicative separability of the marginal rate of technical substitution (MRTS) between effort and skill. This identifies the additive contest as the natural representative of the class and shows that the large body of results built on it applies far beyond the additive specification itself.

Second, strategic equivalence does not render the production technology irrelevant. The transformations generating the equivalence class preserve only ordinal information (rankings of outputs) whereas contest design depends on cardinal features of the environment, such as the hazard rate of the skill distribution, which change under these transformations. Holding the skill distribution fixed, strategically equivalent technologies may therefore call for different and even opposite prize structures.

When characterizing the equivalence class, we show that a contest is strategically equivalent to an additive contest if and only if its production function admits a strictly increasing transformation that is additively separable. Whenever two production technologies are related in this way, the resulting contests are the same game up to a relabeling of strategies, so that all pure-strategy equilibria correspond, also with heterogeneous or risk-averse players. Conversely, within the symmetric framework, the relation is not only sufficient but necessary: if first-order behavior corresponds across all skill distributions and cost functions, the technologies must be so related. This characterization establishes a close connection between the theory of strategically equivalent contests and the classical literature on additive utility representations (e.g., Leontief, 1947, Samuelson, 1947, Debreu, 1960, Goldman and Uzawa, 1964, Gorman, 1968). The central question in that literature is whether preferences admit an additively separable representation after a strictly increasing transformation of the utility function. We find that characterizing strategically equivalent contests is mathematically identical to characterizing utility functions that admit an additively separable representation. In both settings, the relevant transformations preserve only ordinal information, i.e., rankings of consumption bundles in the case of preferences and rankings of contestants' performances in the case of contests. Consequently, the classical characterization of additive representations carries over directly to our study of contests.

Building on the findings of Samuelson (1947), we then show that a contest with a general production technology is strategically equivalent to the additive benchmark if and only if the MRTS is multiplicatively separable. The characterization is not confined to one-dimensional effort. When players choose multiple efforts, a contest is strategically equivalent to the additive benchmark if and only if the MRTS between every pair of variables is multiplicatively separable.

The characterization theorem also provides a simple way of transferring existing results obtained for additive contests to the entire equivalence class. We develop these implications in two applications (which are illustrative rather than exhaustive). First, we study optimal contest design. Optimal prize structures have been studied extensively in contests and tournaments (e.g., Krishna and Morgan, 1998; Moldovanu and Sela, 2001; Akerlof and Holden, 2012). We build on recent results by Drugov and Ryvkin (2020b), who analyze optimal prize structures in contests with additive production technologies. By our equivalence theorem, their results immediately extend to all production technologies whose MRTS is multiplicatively separable. The extension operates on the transformed skill distribution, however, which is why the optimal prize structure can differ across technologies within the class. We then investigate how properties of the MRTS affect contest design.

In particular, we show that the monotonicity of the MRTS in skill determines which prize structures are optimal. When all prize structures that award nothing to the lowest rank are equivalent in the additive benchmark, an MRTS that increases in skill makes the winner-takes-all contest optimal, whereas an MRTS that decreases in skill makes the loser-gets-nothing contest optimal. More generally, the optimal prize structure depends on the monotonicity of the product of the skill component of the MRTS and the hazard rate of the skill distribution.

Second, we show that comparative-static results on the effect of noise also extend directly to the entire equivalence class. Building on Drugov and Ryvkin (2020a), we show that the relevant stochastic-order and entropy characterizations remain valid after replacing the original skill variable by its transformed counterpart induced by the production technology.

Our paper is most closely related to the literature studying strategically equivalent contests. A relatively early contribution is Baye and Hoppe (2003), who demonstrate an equivalence between certain types of rent-seeking contests, innovation tournaments, and patent-race games. Several papers derive strategic equivalence between Lazear–Rosen tournaments and ratio-form contests under specific distributional and technological assumptions (e.g., Jia, 2008; Fu and Lu, 2012, Ryvkin and Drugov, 2020). These results imply that particular non-additive production technologies (e.g., multiplicative) can be transformed into additive representations. In contrast, we provide a complete characterization of the production technologies that generate contests strategically equivalent to the additive Lazear–Rosen tournament.

In two of our earlier papers (Bastani et al., 2022, Giebe and Gürtler, 2026), we study contests with general production functions, but we do not try to characterize strategically equivalent contests. There also exist some papers that use the so-called Mirrlees formulation rather than a state-space approach to model the contest (e.g., Koh, 1992 and Kirkegaard, 2025). Under the Mirrlees formulation, the primitive object is the distribution of output conditional on effort. Because this object jointly reflects the production technology and the underlying uncertainty, it is generally difficult to isolate the role of the production technology itself. By contrast, the state-space approach adopted in this paper allows us to characterize strategic equivalence directly in terms of properties of the production function.

The remainder of the paper is organized as follows. Section 2 introduces the model and characterizes the equilibrium. Section 3 characterizes the class of production technologies strategically equivalent to the additive benchmark. Section 4 applies the characterization to optimal contest design. Section 5 extends comparative-static results on the effect of noise to the entire equivalence class. Section 6 concludes. All proofs are relegated to the Appendix.

2. Model and equilibrium characterization

As explained in the introduction, our goal is to characterize strategically equivalent contests. In this section, we describe one representative contest. The general assumptions that we introduce hold for all considered contests, while the specifics (such as the exact production and cost functions or the sets E and Θ to be introduced below) may vary.

We consider a Lazear-Rosen tournament with $n \geq 2$ risk-neutral players who compete for n prizes that are ordered as $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ (with at least one strict inequality between prizes). All players $i \in \{1, \ldots, n\} =: \mathcal{N}$ simultaneously choose effort $e_i \geq \underline{e}$, and the cost of effort $c(e_i)$ is described by a twice continuously differentiable and strictly increasing function satisfying $c(\underline{e}) \leq w_n$ and $\sup_{e \geq \underline{e}} c(e) > w_1 + c(\underline{e})$.[3] The first condition ensures that participation is individually rational, the second guarantees the existence of $\overline{e} > \underline{e}$ with $c(\overline{e}) = w_1 + c(\underline{e})$. Any effort $e > \overline{e}$ yields an expected payoff of at most $w_1 - c(e) < -c(\underline{e})$, whereas choosing $\underline{e}$ guarantees at least $w_n - c(\underline{e}) \geq -c(\underline{e})$. Efforts above $\overline{e}$ are therefore strictly dominated, and no player would ever choose one. Define $E := [\underline{e}, \overline{e}]$.[4]

Apart from effort, player i's output in the contest depends on the realization θ_i of a random variable Θ_i. We refer to the random variable as a player's skill, but we could alternatively interpret it as luck or noise. The realization θ_i is unknown to all players, including player i. It is commonly known, however, that Θ_i is independently and absolutely continuously distributed according to the cdf F and pdf f. We assume that f has convex support denoted by $\Theta = \{x : f(x) > 0\}$, with lower bound $\underline{\theta} \in \mathbb{R} \cup \{-\infty\}$ and upper bound $\overline{\theta} \in \mathbb{R} \cup \{+\infty\}$. We call such a distribution *admissible* if, in addition, the weighted integrals defining the marginal rank probabilities below are finite and may be differentiated under the integral sign.

Player i's output (or performance) is given by the three times continuously differentiable production function $g(e_i, \theta_i)$. We assume that $\frac{\partial g}{\partial \theta_i} > 0$ and $\frac{\partial g}{\partial e_i} > 0$ for all $(e_i, \theta_i) \in E \times \Theta$, so that the following ratio is well defined and strictly positive. We denote by

$$\mathrm{MRTS}\,(e_i, \theta_i) = \frac{\frac{\partial g}{\partial e_i}\,(e_i, \theta_i)}{\frac{\partial g}{\partial \theta_i}\,(e_i, \theta_i)}$$

the marginal rate of technical substitution (MRTS). Prizes are allocated according to the ranking of outputs. That is, the player with the jth largest output receives prize w_j (for

[3] Throughout the paper, "increasing" means nondecreasing and "decreasing" means nonincreasing; we add "strictly" whenever the distinction matters. Furthermore, most models on the Lazear–Rosen tournament assume that the cost function is strictly convex. To characterize strategically equivalent contests, we transform the cost function and not all transformations satisfy convexity. That is why we do not impose this assumption.

[4] In most contest models it is assumed that $\underline{e} = 0$. However, as we transform variables to obtain strategically equivalent contests, $\underline{e}$ can be different from zero in our model.

$j \in \{1, \ldots, n\}$). Ties are broken randomly.[5]

We adopt a first-order approach and characterize the equilibrium through the players' first-order conditions to their maximization problems. We restrict attention to interior symmetric equilibria, at which the first-order condition holds with equality. In the case of an additive production function, Drugov and Ryvkin (2020b) have derived conditions under which the first-order approach is valid. The conditions require the random variable Θ_i to be sufficiently dispersed and/or the cost function sufficiently convex. In Sections 4.1 and 5, we consider contests which are strategically equivalent to the additive one and we thus refer to the conditions derived by Drugov and Ryvkin (2020b). In Section 4.2, we consider other types of production functions outside the equivalence class, and we maintain the validity of the first-order approach as an assumption.

Assume that all players other than one considered player choose effort e^*. The considered player's payoff from choosing effort e is given by

$$u(e, e^*) = \sum_{j=1}^{n} p^{(j)}(e, e^*) w_j - c(e),$$

where $p^{(j)}(e, e^*)$ denotes the probability that the considered player's output is ranked j. Define $g_e : \Theta \to \mathbb{R}$ by $g_e(x) = g(e, x)$ and denote its inverse on the image $g_e(\Theta)$ by g_e^{-1}. For values outside this image (which may arise for off-equilibrium efforts if Θ is bounded), we set $g_e^{-1}(v) := \underline{\theta}$ if $v \leq \inf g_e(\Theta)$ and $g_e^{-1}(v) := \overline{\theta}$ if $v \geq \sup g_e(\Theta)$, so that $F(g_{e^*}^{-1}(g(e,x)))$ evaluates to 0 or 1, respectively, in the rank probabilities below. Player i then performs better than k iff

$$g_{e_k}^{-1}(g(e_i, \theta_i)) > \theta_k.$$

The probability $p^{(j)}(e, e^*)$ can thus be written as

$$p^{(j)}(e, e^*) = \binom{n-1}{j-1} \int F\left(g_{e^*}^{-1}(g(e, x))\right)^{n-j} \left(1 - F\left(g_{e^*}^{-1}(g(e, x))\right)\right)^{j-1} dF(x).$$

[5] Note that, since g is strictly increasing in θ_i and the random variables Θ_i are independently and absolutely continuously distributed, the distribution of each player's output is atomless, so that ties occur with probability zero.

In a symmetric equilibrium, the first-order condition is

$$\left.\frac{\partial}{\partial e}\left(u\left(e,e^*\right)\right)\right|_{e=e^*}=\sum_{j=1}^{n}\beta_j\left(e^*\right)w_j-c'\left(e^*\right)=0,$$

where

$$\begin{aligned}\beta_j\left(e^*\right)=&\left.\frac{\partial}{\partial e}\left(p^{(j)}\left(e,e^*\right)\right)\right|_{e=e^*}\\=&\binom{n-1}{j-1}\int\left(\begin{array}{c}(n-j)F\left(g_{e^*}^{-1}\left(g(e^*,x)\right)\right)^{n-j-1}f\left(g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)\right)\cdot\\ \frac{\partial}{\partial e}\left(g_{e^*}^{-1}(g(e,x))\right)\big|_{e=e^*}\left(1-F\left(g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)\right)\right)^{j-1}\\ -(j-1)F\left(g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)\right)^{n-j}\left(1-F\left(g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)\right)\right)^{j-2}\cdot\\ f\left(g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)\right)\frac{\partial}{\partial e}\left(g_{e^*}^{-1}(g(e,x))\right)\big|_{e=e^*}\end{array}\right)dF\left(x\right).\end{aligned}$$

Note first that $g_{e^*}^{-1}\left(g\left(e^*,x\right)\right)=g_{e^*}^{-1}\left(g_{e^*}\left(x\right)\right)=x$. Notice further that $\frac{\partial}{\partial e}\left(g_{e^*}^{-1}(g(e,x))\right)\big|_{e=e^*}=\frac{\partial g(e^*,x)}{\partial e^*}/\frac{\partial g(e^*,x)}{\partial x}=\mathrm{MRTS}\left(e^*,x\right)$ (see Bastani et al., 2022, p.268).

The equilibrium marginal probability of reaching rank j can thus be rewritten as[6]

$$\beta_j(e^*)=\binom{n-1}{j-1}\int\mathrm{MRTS}(e^*,x)F(x)^{n-j-1}(1-F(x))^{j-2}(n-j-(n-1)F(x))f(x)\,dF(x)$$

or

$$\beta_j(e^*)=\binom{n-1}{j-1}\int\mathrm{MRTS}(e^*,x)F(x)^{n-j-1}(1-F(x))^{j-2}(n-j-(n-1)F(x))f(x)^2\,dx$$

Define

$$\gamma_j(x)=\binom{n-1}{j-1}F(x)^{n-j-1}(1-F(x))^{j-2}(n-j-(n-1)F(x))f(x)^2$$

so that

$$\beta_j(e^*)=\int\mathrm{MRTS}(e^*,x)\gamma_j(x)\,dx.$$

For the endpoint ranks $j=1$ and $j=n$, the expression for γ_j is to be read after cancellation: the factor $n-j-(n-1)F(x)$ equals $(n-1)(1-F(x))$ for $j=1$ and $-(n-1)F(x)$ for $j=n$, so that $\gamma_1(x)=(n-1)F(x)^{n-2}f(x)^2$ and $\gamma_n(x)=-(n-1)(1-F(x))^{n-2}f(x)^2$. In particular, no negative powers of F or $1-F$ remain.

[6] We use $F(x)^k$ to denote $(F(x))^k$ and, similarly, $f(x)^k$ to denote $(f(x))^k$.

3. Strategically equivalent contests

Consider two distinct contests that differ in their production technology.[7] Suppose that the equilibrium of the first contest has been characterized, and let $e(w_1, \ldots, w_n, F, c)$ denote the (symmetric) equilibrium effort. In some instances, the equilibrium effort of the second contest can be deduced from that of the first contest through monotone transformations. Specifically, there are instances in which, letting $\tilde{e}(w_1, \ldots, w_n, F, c)$ denote the equilibrium effort of the second contest, $\tilde{e}(w_1, \ldots, w_n, F, c) = \alpha(e(w_1, \ldots, w_n, F \circ \beta, c \circ \alpha))$ for appropriately chosen strictly increasing functions α and β. Using $\tilde{F} = F \circ \beta$ and $\tilde{c} = c \circ \alpha$, the condition can be restated as $e(w_1, \ldots, w_n, \tilde{F}, \tilde{c}) = \alpha^{-1}(\tilde{e}(w_1, \ldots, w_n, \tilde{F} \circ \beta^{-1}, \tilde{c} \circ \alpha^{-1}))$, implying that equilibrium efforts can be deduced in either direction. In the following, we refer to two contests whose equilibrium efforts are related in this way as *strategically equivalent contests*; a formal definition is provided below. Note that the transformations α and β are required to be fixed functions, meaning that they may depend on the production function, but not on the primitives $(w_1, \ldots, w_n, F, c)$. This uniformity is essential: a single equilibrium effort level could always be matched by some increasing transformation, so that the notion of strategic equivalence would be meaningless if the transformations were allowed to vary with the environment. We return to this point in Section 3.1.

To provide a more systematic analysis of the strategic equivalence between contests, let the two considered contest's production functions be given by $g_1(e_i, \theta_i)$ and $g_2(e_i, \theta_i)$. We will show that the two contests are strategically equivalent if strictly increasing functions ϕ, A, and B exist such that $\phi(g_1(e_i, \theta_i)) = g_2(A(e_i), B(\theta_i))$ for all $e_i \in E$ and $\theta_i \in \Theta$. In particular, if the first contest has skill distribution F and cost function c, and the second has skill distribution $\tilde{F} = F \circ B^{-1}$ and cost function $\tilde{c} = c \circ A^{-1}$, then, for every strategy profile $(e_1, \ldots, e_n)$, the players' expected payoffs in the two contests (u_i and $\tilde{u}_i$) satisfy

$$\tilde{u}_i(A(e_1), \ldots, A(e_n)) = u_i(e_1, \ldots, e_n) \quad \text{for all } i \in \mathcal{N}.$$

As a consequence, $(e_1^*, \ldots, e_n^*)$ is an equilibrium of the first contest if and only if $(A(e_1^*), \ldots, A(e_n^*))$ is an equilibrium of the second contest.

[7] As will become clear below, relating the two contests also involves suitably transformed skill distributions and cost functions.

To understand this, fix a strategy profile $(e_1, \ldots, e_n)$ and a vector of skill realizations $(\theta_1, \ldots, \theta_n)$. Since ϕ is strictly increasing and $\phi(g_1(e_i, \theta_i)) = g_2(A(e_i), B(\theta_i))$ for all i, the ranking of the outputs $g_2(A(e_i), B(\theta_i))$ across players coincides with the ranking of the outputs $g_1(e_i, \theta_i)$ state by state. Moreover, the random variable $B(\Theta_i)$ is distributed according to $\tilde{F} = F \circ B^{-1}$ (as $B(\Theta_i) \leq t \Leftrightarrow \Theta_i \leq B^{-1}(t)$). Hence, for every rank j and every player i, the probability that player i's output is ranked j in the second contest under the profile $(A(e_1), \ldots, A(e_n))$ and skill distribution $\tilde{F}$ equals the corresponding probability in the first contest under $(e_1, \ldots, e_n)$ and F. Since, in addition, $\tilde{c}(A(e_i)) = c(A^{-1}(A(e_i))) = c(e_i)$, the expected payoffs coincide.

Regarding the equilibria of the two contests, notice that A is a bijection from E onto $A(E)$, so that the strategy spaces of the two contests are in one-to-one correspondence. By the payoff identity, a profitable deviation from a profile in one contest translates into a profitable deviation from the corresponding profile in the other contest, and vice versa. Stated differently, if players do not wish to deviate from a strategy profile in one of the contests, there is no profitable deviation from the (transformed) strategy profile in the other contest.

This argument has two features. First, it operates at the level of the normal-form game: expected payoffs coincide profile by profile, so that the two contests represent the same game up to a relabeling of strategies. In particular, *all* pure-strategy equilibria correspond, not merely the symmetric one, and the first-order approach plays no role in the equivalence itself. Second, neither symmetry nor risk neutrality is needed. The argument applies without change if players are heterogeneous, i.e., if skills are distributed according to player-specific distributions F_i (as, e.g., in Giebe and Gürtler, 2026) and effort costs are given by player-specific functions c_i. It further extends to arbitrary preferences over prizes and effort, i.e., to expected payoffs of the form $\sum_{j=1}^{n} p^{(j)} U(w_j, e_i)$. To see this, notice that the transformation leaves the rank probabilities unchanged profile by profile, and the transformed preferences $\tilde{U}(w_j, \tilde{e}_i) := U(w_j, A^{-1}(\tilde{e}_i))$ satisfy $\tilde{U}(w_j, A(e_i)) = U(w_j, e_i)$, so that expected payoffs again coincide. This covers risk aversion over the net payoff, $U(w_j, e_i) = v(w_j - c(e_i))$ with v strictly increasing and concave, as well as the additively separable case $U(w_j, e_i) = v(w_j) - c(e_i)$. The symmetric framework of Section 2 is thus required for the necessity result of Section 3.1

and for the applications, but not for the equivalence itself.

Summing up, we have shown that the two contests are strategically equivalent if strictly increasing functions ϕ, A, and B exist such that $\phi\left(g_1\left(e_i, \theta_i\right)\right) = g_2\left(A\left(e_i\right), B\left(\theta_i\right)\right)$ for all $e_i \in E$ and $\theta_i \in \Theta$. This condition underlying the equivalence of contests can be understood as rank preservation under transformation, i.e., for any given efforts and skill realizations, the ranking of outputs remains unchanged after the transformation. Two notions are thus in play: the behavioral notion of corresponding equilibrium efforts, with which we opened this section, and the transformation condition just derived. The condition implies the behavioral notion; Section 3.1 establishes the converse at the level of symmetric first-order behavior, so that the two notions coincide and the transformation condition can serve as the formal definition of strategically equivalent contests. For ease of notation, we denote a contest with production function $g_t(e_i, \theta_i)$ by g_t^c.

Definition 1. *A contest g_1^c is strategically equivalent to a contest g_2^c, written as $g_1^c \sim g_2^c$, if strictly increasing functions ϕ, A, and B exist such that $\phi\left(g_1\left(e_i, \theta_i\right)\right) = g_2\left(A\left(e_i\right), B\left(\theta_i\right)\right)$ for all $e_i \in E$ and $\theta_i \in \Theta$.*

Here, contest g_2^c is understood to have effort set $A(E)$ and skill support $B(\Theta)$, in line with the observation from Section 2 that these sets may vary across the considered contests.[8] It is easy to show that the relation introduced in Definition 1 represents an equivalence relation.

Lemma 1. *The relation introduced in Definition 1 satisfies reflexivity, symmetry, and transitivity, and is thus an equivalence relation.*

Most of the literature on the Lazear–Rosen tournament assumes an additive production function of the form $g(e_i, \theta_i) = e_i + \theta_i$. We are thus particularly interested in contests that are equivalent to the additive contest since the existing results from the contest literature could then be readily applied to other contest types. We denote the contest with an additive production function of the form $g(e_i, \theta_i) = e_i + \theta_i$ by g_{add}^c and the equivalence class by $[g_{add}^c]$.

[8]In what follows, whenever differentiability of the transformations is required (Theorems 1 and 2, Corollary 1, and Proposition 1), ϕ, A, and B are understood to be continuously differentiable with strictly positive derivatives.

According to Definition 1, a contest with a general production function $g(e_i, \theta_i)$ is equivalent to the additive contest if a monotone transformation ϕ of $g(e_i, \theta_i)$ exists such that the resulting expression is additively separable in effort and skill. Hence, the characterization of contests that are strategically equivalent to the additive one boils down to finding the production functions that can be monotonically transformed into an additive function. In the context of utility functions, Samuelson (1947, p.175–176) has solved a mathematically equivalent problem, and the proof of the following theorem makes use of his results.

Theorem 1. *For a contest g^c, we observe $g^c \in [g^c_{add}]$ if and only if there are strictly positive functions $q(\theta_i)$ and $r(e_i)$ (with q being independent of e_i and r of θ_i) such that the MRTS can be written as $\mathrm{MRTS}(e_i, \theta_i) = r(e_i)\, q(\theta_i)$ for all $e_i \in E$ and $\theta_i \in \Theta$ (i.e., $\mathrm{MRTS}(e_i, \theta_i)$ is multiplicatively separable in e_i and θ_i).*

The theorem tells us that any contest with a production function that has a multiplicatively separable MRTS is strategically equivalent to the additive contest. Interestingly, most standard production functions such as the Cobb-Douglas function, the CES function, or the quasilinear function possess a multiplicatively separable MRTS. The same is true for a production function of the form $g(e_i, \theta_i) = e_i + \theta_i + e_i \cdot \theta_i$, which combines the additive and multiplicative case. An example of a function with a non-separable MRTS is given by $g(e_i, \theta_i) = e_i + \theta_i + e_i \cdot \theta_i^2$.

Moreover, for contests in the equivalence class, the transformation into the additive benchmark can be constructed explicitly from the MRTS.

Corollary 1. *Let $\mathrm{MRTS}(e_i, \theta_i) = r(e_i) q(\theta_i)$ with $r, q > 0$. Then $g^c \in [g^c_{add}]$ via the transformations*

$$A(e_i) = \int^{e_i} r(s)\, ds \qquad \text{and} \qquad B(\theta_i) = \int^{\theta_i} \frac{1}{q(s)}\, ds,$$

i.e., there exists a strictly increasing function ϕ such that $\phi(g(e_i, \theta_i)) = A(e_i) + B(\theta_i)$ for all $e_i \in E$ and $\theta_i \in \Theta$.

The corollary turns the characterization into a simple recipe. For the multiplicative production function $g(e_i, \theta_i) = e_i \theta_i$ considered in Example 1 below, we have $\mathrm{MRTS}(e_i, \theta_i) = \theta_i / e_i$, so that $r(e_i) = 1/e_i$ and $q(\theta_i) = \theta_i$, and the corollary delivers the transformation used in the example: $A(e_i) = \ln e_i$ and $B(\theta_i) = \ln \theta_i$. Similarly, for the Cobb–Douglas function $g(e_i, \theta_i) = e_i^\alpha \theta_i^\beta$ (with $\alpha, \beta > 0$), the corollary yields $A(e_i) = \alpha \ln e_i$ and $B(\theta_i) = \beta \ln \theta_i$. The recipe also handles less familiar members of the class: for $g(e_i, \theta_i) = e_i + \theta_i + e_i\theta_i$ (with $e_i, \theta_i > 0$), we have $\mathrm{MRTS}(e_i, \theta_i) = (1 + \theta_i)/(1 + e_i)$, so that the corollary yields $A(e_i) = \ln(1 + e_i)$ and $B(\theta_i) = \ln(1 + \theta_i)$; the associated transformation is $\phi(v) = \ln(1 + v)$, since $(1 + e_i)(1 + \theta_i) = 1 + g(e_i, \theta_i)$.[9]

We conclude this section with an illustrative example.

Example 1. *Consider a contest between three players with additive production function $g(e, \theta) = e + \theta$. Supposing that the other two players choose the symmetric effort e, player i maximizes*

$$\begin{aligned}
&w_1 \int F(\theta_i + e_i - e)^2 f(\theta_i) d\theta_i + w_2 \int 2F(\theta_i + e_i - e)(1 - F(\theta_i + e_i - e)) f(\theta_i) d\theta_i \\
&+ w_3 \int (1 - F(\theta_i + e_i - e))^2 f(\theta_i) d\theta_i - c(e_i) \\
=&(w_1 - 2w_2 + w_3) \int F(\theta_i + e_i - e)^2 f(\theta_i) d\theta_i + 2(w_2 - w_3) \int F(\theta_i + e_i - e) f(\theta_i) d\theta_i + w_3 - c(e_i).
\end{aligned}$$

The first-order condition, evaluated at symmetric effort e is

$$(w_1 - 2w_2 + w_3) \int 2F(\theta_i) f(\theta_i)^2 d\theta_i + 2(w_2 - w_3) \int f(\theta_i)^2 d\theta_i = c'(e). \tag{1}$$

Now consider a contest between three players with multiplicative production function $g(e, \theta) = e\theta$, where we assume $\Theta \subset (0, \infty)$ and $\underline{e} > 0$, so that outputs are positive and the logarithmic transformation below is well defined.[10] *Note that the MRTS is multiplicatively separable:* $\mathrm{MRTS}(e, \theta) = \theta/e$. *Supposing that the other two players choose the symmetric*

[9] The class also contains technologies from applied work. Suppose that a contestant's output is distributed according to G^{a+e_i}, i.e., the distribution of the best of $a + e_i$ draws from a distribution G, where $a > 0$ is an ability parameter, as, e.g., in the innovation contests of Fullerton and McAfee (1999) and Giebe (2014). This output can be written as $g(e_i, \theta_i) = G^{-1}\left(\theta_i^{1/(a+e_i)}\right)$ with skill Θ_i uniformly distributed on $(0, 1)$. The transformation $\phi(v) = -\ln(-\ln G(v))$ yields $\phi(g(e_i, \theta_i)) = \ln(a + e_i) + B(\theta_i)$ with $B(\theta_i) = -\ln(-\ln \theta_i)$, so that $g^c \in [g^c_{add}]$.

[10] The computation is straightforward and the multiplicative case is well known. See, in particular, Ryvkin and Drugov (2020, Section 2.4), whose transformed cost function $c_m(e) = \int_0^e c'(t)\, t\, dt$ is our $\tilde{c} = c \circ A^{-1}$ for $A = \ln$. We work it through here for illustration purposes.

effort e, player i maximizes

$$
\begin{aligned}
&w_1 \int F\left(\frac{\theta_i e_i}{e}\right)^2 f(\theta_i) d\theta_i + w_2 \int 2F\left(\frac{\theta_i e_i}{e}\right)\left(1 - F\left(\frac{\theta_i e_i}{e}\right)\right) f(\theta_i) d\theta_i \\
&+ w_3 \int \left(1 - F\left(\frac{\theta_i e_i}{e}\right)\right)^2 f(\theta_i) d\theta_i - c(e_i) \\
=&(w_1 - 2w_2 + w_3) \int F\left(\frac{\theta_i e_i}{e}\right)^2 f(\theta_i) d\theta_i + 2(w_2 - w_3) \int F\left(\frac{\theta_i e_i}{e}\right) f(\theta_i) d\theta_i + w_3 - c(e_i).
\end{aligned}
$$

The first-order condition, evaluated at symmetric effort e is

$$
(w_1 - 2w_2 + w_3) \int 2F(\theta_i) \frac{\theta_i}{e} f(\theta_i)^2 d\theta_i + 2(w_2 - w_3) \int \frac{\theta_i}{e} f(\theta_i)^2 d\theta_i = c'(e),
$$

which we can write as

$$
(w_1 - 2w_2 + w_3) \int 2F(\theta_i) f(\theta_i)^2 \theta_i d\theta_i + 2(w_2 - w_3) \int f(\theta_i)^2 \theta_i d\theta_i = ec'(e). \tag{2}
$$

We now demonstrate how to use a strategically equivalent additive contest to solve the above contest with multiplicative production function. The event of player i winning against player j can be written as

$$
e_i \theta_i > e_j \theta_j \iff \ln(e_i) + \ln(\theta_i) > \ln(e_j) + \ln(\theta_j) \iff \tilde{e}_i + \tilde{\theta}_i > \tilde{e}_j + \tilde{\theta}_j,
$$

where we have defined $\tilde{e}_s := \ln(e_s)$ and $\tilde{\theta}_s := \ln(\theta_s)$ (for $s \in \{i, j\}$) such that $e_s = \exp(\tilde{e}_s)$ and $\theta_s = \exp(\tilde{\theta}_s)$. We now use the first-order condition of the additive contest, (1), as a template and apply it to our transformed contest, using distribution $\tilde{F}(x) = F(\exp(x))$ and cost function $\tilde{c}(\tilde{e}) = c(\exp(\tilde{e}))$. Note that

$$
\tilde{f}(\tilde{\theta}_i) = \frac{dF(\exp(\tilde{\theta}_i))}{d\tilde{\theta}_i} = f(\exp(\tilde{\theta}_i)) \exp(\tilde{\theta}_i) \text{ and } \frac{d\tilde{c}(\tilde{e})}{d\tilde{e}} = c'(\exp(\tilde{e})) \exp(\tilde{e}).
$$

Inserting everything in (1), we get the first-order condition of the transformed contest

$$
(w_1 - 2w_2 + w_3) \int 2\tilde{F}(\tilde{\theta}_i) \tilde{f}(\tilde{\theta}_i)^2 d\tilde{\theta}_i + 2(w_2 - w_3) \int \tilde{f}(\tilde{\theta}_i)^2 d\tilde{\theta}_i = \tilde{c}'(\tilde{e}).
$$

Substituting the above definitions, this becomes

$$
\begin{aligned}
&(w_1 - 2w_2 + w_3) \int 2F(\exp(\tilde{\theta}_i)) \left(f(\exp(\tilde{\theta}_i)) \exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i \\
&+ 2(w_2 - w_3) \int \left(f(\exp(\tilde{\theta}_i)) \exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i = c'(\exp(\tilde{e})) \exp(\tilde{e}),
\end{aligned}
$$

characterizing the equilibrium effort $\tilde{e}$ *of the transformed contest. Applying a change of variables on the LHS,* $\theta_i = \exp(\tilde{\theta}_i)$*, implying* $d\theta_i/d\tilde{\theta}_i = \exp(\tilde{\theta}_i) \Rightarrow d\tilde{\theta}_i = d\theta_i/\theta_i$ *we get*

$$\left(f(\exp(\tilde{\theta}_i))\exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i = f(\theta_i)^2\theta_i^2\frac{d\theta_i}{\theta_i} = f(\theta_i)^2\theta_i d\theta_i,$$

and therefore the first-order condition becomes

$$\begin{aligned}(w_1 - 2w_2 + w_3)&\int 2F(\theta_i)f(\theta_i)^2\theta_i d\theta_i \\ + 2(w_2 - w_3)&\int f(\theta_i)^2\theta_i d\theta_i = c'(\exp(\tilde{e}))\exp(\tilde{e}).\end{aligned}$$

As the LHS is identical to the LHS of the original contest's first-order condition, (2)*, the two RHS coincide under the transformation:*

$$ec'(e) = c'(\exp(\tilde{e}))\exp(\tilde{e}).$$

Since $e = \exp(\tilde{e})$ *under the transformation, the transformed first-order condition coincides with the original first-order condition. Hence the equilibrium of the transformed (additive) contest maps back exactly to the equilibrium of the original (multiplicative) contest.*

3.1. Necessity: a behavioral foundation of Definition 1

At the beginning of Section 3, we have shown that rank preservation under transformation is sufficient for two contests to be strategically equivalent, and we have thus used this condition in our Definition 1 of strategically equivalent contests. It is natural to ask whether the condition is also necessary: could two contests generate corresponding equilibrium efforts in all environments although their technologies are not related as in Definition 1? The following result shows that the answer is negative at the level of symmetric first-order behavior: whenever the symmetric first-order conditions of the two contests have corresponding solutions in all environments, the technologies must be related as in Definition 1. This shows that the transformation condition of Definition 1 is exactly the condition pinned down by behavior, with sufficiency read at the level of equilibria and necessity at the level of symmetric first-order conditions. Necessity obtains already from first-order behavior in two-player contests with a single fixed prize gap; all the identifying power comes from jointly varying the skill distribution and the cost function.

Proposition 1. *Let A and B be continuously differentiable with strictly positive derivatives, and let contest 2 be defined on the effort set $A(E)$ and the skill support $B(\Theta)$. Suppose that, for $n = 2$, some fixed prizes $w_1 > w_2 \geq 0$, every admissible skill distribution F with support Θ, and every admissible cost function c (i.e., every cost function satisfying the assumptions of Section 2)*[11]*, first-order behavior in the two contests corresponds, i.e., e^* solves the symmetric first-order condition of contest 1 under (F, c) if and only if $A(e^*)$ solves the symmetric first-order condition of contest 2 under $(F \circ B^{-1}, c \circ A^{-1})$. Then $g_1^c \sim g_2^c$; specifically, there exists a strictly increasing function ϕ such that $\phi(g_1(e_i, \theta_i)) = g_2(A(e_i), B(\theta_i))$ for all $e_i \in E$ and $\theta_i \in \Theta$.*

Proposition 1 requires that behavior corresponds for *every* skill distribution, and this requirement cannot be dropped: if the two contests generate corresponding first-order behavior only for one particular skill distribution, their technologies need not be related as in Definition 1.[12] Varying the skill distribution is not the only way to identify the technologies, however. If the skill distribution is held fixed (with full support) but the correspondence of first-order behavior is required for all admissible cost functions, all numbers of players $n \geq 2$, and all prize vectors, the conclusion of Proposition 1 continues to hold.

To summarize, the equivalence class $[g_{add}^c]$ is simultaneously a technological, a representational, and a behavioral object: At the level of the technology, the MRTS is multiplicatively separable (Theorem 1), a condition directly checkable from g. At the level of the representation, the technology admits an ordinal additive representation $\phi(g(e_i, \theta_i)) = A(e_i) + B(\theta_i)$ (Definition 1), with the transformations obtainable in closed form from the MRTS (Corollary

[11] The effort set E of contest 1 is held fixed throughout this proposition. While in Section 2 the upper bound $\bar{e}$ is derived from (c, w_1), the argument operates entirely at the level of first-order conditions on E, so that the upper bound induced by any particular cost function plays no role.

[12] For an example, let contest 1 be additive, $g_1(e_i, \theta_i) = e_i + \theta_i$, let $A = B = id$, and let contest 2 be quasilinear, $g_2(e_i, \theta_i) = e_i + t(\theta_i)$, with the slope of t chosen such that $\text{MRTS}_2(e_i, \theta_i) = 1 + \kappa(\theta_i)$, where κ is twice continuously differentiable with $|\kappa| < 1$, $\kappa \not\equiv 0$, and $\int_0^1 \kappa(x)\,dx = 0$. If F is the uniform distribution on $[0, 1]$, the two symmetric first-order conditions coincide for $n = 2$ and every admissible cost function, because $\int_0^1 \text{MRTS}_2(e, x)\, f(x)^2\,dx = 1 + \int_0^1 \kappa(x)\,dx = 1$, exactly as in the additive contest. Still, no strictly increasing ϕ with $\phi(g_1(e_i, \theta_i)) = g_2(e_i, \theta_i)$ exists: the left-hand side is constant whenever $e_i + \theta_i$ is constant, whereas the right-hand side is not.

1). At the level of behavior, the contest is strategically equivalent to the additive benchmark: the transformation condition implies that the two contests are the same game up to a relabeling of strategies, so that all pure-strategy equilibria correspond, also with heterogeneous players and arbitrary preferences over prizes and effort, while, conversely, correspondence of symmetric first-order behavior across environments already forces the transformation condition (Proposition 1). Definition 1 is therefore not a modeling convenience: it is the only notion of equivalence consistent with all three descriptions.

3.2. Extension: multiple efforts

So far, we have assumed that efforts are one-dimensional. There are many contest applications, however, where players choose multiple efforts (as, e.g., in Holmström and Milgrom, 1991). For instance, there is a large literature on sabotage in contests (e.g., Lazear, 1989, Chowdhury and Gürtler, 2015), which assumes that players choose productive effort to increase their performance, but also destructive effort to lower other players' performances.[13]

In this extension, we allow for players' efforts to be multi-dimensional, and we denote player i's vector of efforts by $\mathbf{e}_i = (e_{i1}, \ldots, e_{im})$, where $m \geq 2$ denotes the number of efforts. The corresponding output is given by $g(\mathbf{e}_i, \theta_i)$. Definition 1 extends naturally to this setting: two contests are strategically equivalent if there are strictly increasing functions ϕ, $A^1, \ldots, A^m$, and B such that $\phi(g_1(\mathbf{e}_i, \theta_i)) = g_2(A^1(e_{i1}), \ldots, A^m(e_{im}), B(\theta_i))$ for all $\mathbf{e}_i$ and θ_i, with the cost function transformed coordinate-wise accordingly. The equivalence class containing the additive contest is again determined by whether a production function can be transformed into an additively separable one. As shown by, e.g., Goldman and Uzawa (1964), this is equivalent to the MRTS between any pair of variables to be independent of all the other variables. As we show in the following theorem, this in turn is equivalent to all MRTS being multiplicatively separable, with each factor being independent of all the other

[13] Considering an additive contest and denoting sabotage levels by s, player i outperforms j if and only if $e_i - s_j + \theta_i \geq e_j - s_i + \theta_j$. This condition is equivalent to $e_i + s_i + \theta_i \geq e_j + s_j + \theta_j$, such that, from the contestants' point of view, a contest with sabotage is equivalent to a contest with two productive efforts. Note that this argument covers untargeted sabotage, where each player l chooses a single sabotage level s_l applied to all opponents; sabotage by third parties then cancels from every pairwise comparison, so the equivalence holds for any n. Targeted sabotage, with victim-specific levels, is not covered.

variables. This generalizes our result from Theorem 1.

Theorem 2. *For a contest g^c, we observe $g^c \in [g^c_{add}]$ if and only if the MRTS between any pair of variables is multiplicatively separable, with the respective functions being independent of all the other variables.*

Example 2. *Consider a contest with three players and production function $g(e_{i1}, e_{i2}, \theta) = (1+e_{i1})(1+e_{i2})\theta = (1+e_{i1}+e_{i2}+e_{i1}e_{i2})\theta$, where we assume $\Theta \subset (0,\infty)$ and nonnegative efforts, so that the logarithmic transformation below is well defined. We demonstrate that the equilibrium effort vector $\mathbf{e}^*$ can be found, equivalently, by solving the contest directly or by solving the strategically equivalent additive contest.*

First note that all three MRTS are multiplicatively separable and independent of the respective third variable (using definition $\mathrm{MRTS}(y_i, y_j) := (\partial g/\partial y_i)/(\partial g/\partial y_j)$*):*

$$\mathrm{MRTS}(e_{i1}, e_{i2}) = \frac{1+e_{i2}}{1+e_{i1}}, \quad \mathrm{MRTS}(e_{i1}, \theta) = \frac{\theta}{1+e_{i1}}, \quad \mathrm{MRTS}(e_{i2}, \theta) = \frac{\theta}{1+e_{i2}}.$$

We first derive the players' first-order conditions in a contest with additive production function $g(e_{i1}, e_{i2}, \theta) = e_{i1} + e_{i2} + \theta$. Player i's objective when choosing efforts e_{i1} and e_{i2} against symmetric efforts $\mathbf{e} = (e_1, e_2)$ of the other two players is given by[14]

$$\begin{aligned}
u_i(e_{i1}, e_{i2}, e_1, e_2) = w_1 \int & F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2)^2 f(\theta_i) d\theta_i \\
& + w_2 \int 2F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2)(1 - F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2)) f(\theta_i) d\theta_i \\
& + w_3 \int (1 - F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2))^2 f(\theta_i) d\theta_i - c(e_{i1}, e_{i2}) \\
&= (w_1 - 2w_2 + w_3) \int F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2)^2 f(\theta_i) d\theta_i \\
& + 2(w_2 - w_3) \int F(\theta_i + e_{i1} + e_{i2} - e_1 - e_2) f(\theta_i) d\theta_i + w_3 - c(e_{i1}, e_{i2}).
\end{aligned}$$

In the following, we will always present only one of the two first-order conditions, the one with respect to effort e_{i1}. Correspondingly, we simply denote by $c_1(e_1, e_2)$ the partial derivative of the cost function with respect to the first argument, evaluated at efforts e_1 and e_2. The

[14]Notice that, in the example from this section, e_1 and e_2 denote the symmetric efforts on the two tasks. In all other sections, the subscript refers to the considered player.

first-order condition with respect to effort e_{i1} *evaluated at the symmetric efforts* e_1 *and* e_2 *is*

$$(w_1 - 2w_2 + w_3) \int 2F(\theta_i) f(\theta_i)^2 d\theta_i + 2(w_2 - w_3) \int f(\theta_i)^2 d\theta_i = c_1(e_1, e_2). \tag{3}$$

Now consider the contest with production function $g(e_{i1}, e_{i2}, \theta) = (1 + e_{i1})(1 + e_{i2})\theta$*. Player* i *wins against another player* j *who chooses efforts* e_1 *and* e_2 *iff*

$$g(e_{i1}, e_{i2}, \theta_i) > g(e_1, e_2, \theta_j) \iff \theta_j < \theta_i \frac{(1 + e_{i1})(1 + e_{i2})}{(1 + e_1)(1 + e_2)}.$$

Therefore, player i*'s objective when playing against the symmetric efforts* e_1 *and* e_2 *of the other two players is given by*

$$\begin{aligned} u_i(e_{i1}, e_{i2}, e_1, e_2) = (w_1 - 2w_2 + w_3) &\int F\left(\theta_i \frac{(1 + e_{i1})(1 + e_{i2})}{(1 + e_1)(1 + e_2)}\right)^2 f(\theta_i) d\theta_i \\ &+ 2(w_2 - w_3) \int F\left(\theta_i \frac{(1 + e_{i1})(1 + e_{i2})}{(1 + e_1)(1 + e_2)}\right) f(\theta_i) d\theta_i + w_3 - c(e_{i1}, e_{i2}). \end{aligned}$$

The first-order condition with respect to effort e_{i1}*, evaluated at the symmetric efforts* e_k*,* $k \in \{1, 2\}$ *is*

$$\begin{aligned} &(w_1 - 2w_2 + w_3) \int \frac{\theta_i}{1 + e_1} 2F(\theta_i) f(\theta_i)^2 d\theta_i + 2(w_2 - w_3) \int \frac{\theta_i}{1 + e_1} f(\theta_i)^2 d\theta_i = c_1(e_1, e_2) \\ \iff &(w_1 - 2w_2 + w_3) \int 2\theta_i F(\theta_i) f(\theta_i)^2 d\theta_i + 2(w_2 - w_3) \int \theta_i f(\theta_i)^2 d\theta_i = (1 + e_1) c_1(e_1, e_2). \end{aligned} \tag{4}$$

We now proceed by using a strategically equivalent additive contest to solve the above analyzed contest with production function $(1 + e_{i1})(1 + e_{i2})\theta$*. Applying the* ln*-function to the event of player* i *winning against another player* j *who chooses efforts* e_1 *and* e_2*, we get*

$$\begin{aligned} &g(e_{i1}, e_{i2}, \theta_i) > g(e_1, e_2, \theta_j) \\ \iff &\ln(1 + e_{i1}) + \ln(1 + e_{i2}) + \ln(\theta_i) > \ln(1 + e_1) + \ln(1 + e_2) + \ln(\theta_j) \\ \iff &\tilde{e}_{i1} + \tilde{e}_{i2} + \tilde{\theta}_i > \tilde{e}_1 + \tilde{e}_2 + \tilde{\theta}_j, \end{aligned}$$

where we have defined $\tilde{e}_s := \ln(1 + e_s)$ *(for* $s \in \{i1, i2, 1, 2\}$*) and* $\tilde{\theta}_t := \ln(\theta_t)$ *(for* $t \in \{i, j\}$*) such that* $e_s = \exp(\tilde{e}_s) - 1$ *and* $\theta_t = \exp(\tilde{\theta}_t)$*. We now use the first-order condition of the additive contest,* (3)*, as a template applied to the transformed additive contest with* *distribution* $\tilde{F}(x) = F(\exp(x))$ *and cost function* $\tilde{c}(\tilde{e}_1, \tilde{e}_2) = c(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1)$*. Note that*

$$\tilde{f}(\tilde{\theta}_i) = \frac{dF(\exp(\tilde{\theta}_i))}{d\tilde{\theta}_i} = f(\exp(\tilde{\theta}_i))\exp(\tilde{\theta}_i)$$

and

$$\frac{\partial \tilde{c}(\tilde{e}_1, \tilde{e}_2)}{\partial \tilde{e}_1} = \frac{\partial c(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1)}{\partial \tilde{e}_1} = c_1(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1)\exp(\tilde{e}_1).$$

Inserting everything in (3)*, we get the first-order condition (with respect to the first effort)*

$$\begin{aligned} &(w_1 - 2w_2 + w_3) \int 2F(\exp(\tilde{\theta}_i)) \left(f(\exp(\tilde{\theta}_i))\exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i \\ &+ 2(w_2 - w_3) \int \left(f(\exp(\tilde{\theta}_i))\exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i = \exp(\tilde{e}_1)c_1(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1), \end{aligned} \tag{5}$$

characterizing the equilibrium efforts $(\tilde{e}_1^*, \tilde{e}_2^*)$ *of the transformed contest.*

Applying a change of variables on the LHS, $\theta_i = \exp(\tilde{\theta}_i)$*, implying* $d\theta_i / d\tilde{\theta}_i = \exp(\tilde{\theta}_i) \Rightarrow d\theta_i = \exp(\tilde{\theta}_i) d\tilde{\theta}_i \Rightarrow d\theta_i = \theta_i d\tilde{\theta}_i$*, we get*

$$\left(f(\exp(\tilde{\theta}_i))\exp(\tilde{\theta}_i)\right)^2 d\tilde{\theta}_i = f(\theta_i)^2 \theta_i^2 \frac{d\theta_i}{\theta_i} = \theta_i f(\theta_i)^2 d\theta_i,$$

and therefore the first-order condition becomes

$$\begin{aligned} &(w_1 - 2w_2 + w_3) \int 2\theta_i F(\theta_i) f(\theta_i)^2 d\theta_i \\ &+ 2(w_2 - w_3) \int \theta_i f(\theta_i)^2 d\theta_i = \exp(\tilde{e}_1)c_1(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1). \end{aligned} \tag{6}$$

As the above LHS is identical to that of the original contest's first-order condition, (4)*, the two RHS coincide under the transformation:*

$$(1 + e_1)c_1(e_1, e_2) = \exp(\tilde{e}_1)c_1(\exp(\tilde{e}_1) - 1, \exp(\tilde{e}_2) - 1). \tag{7}$$

Since $e_k = \exp(\tilde{e}_k) - 1$*,* $k \in \{1, 2\}$*, the equilibrium of the transformed additive contest maps back exactly to the symmetric equilibrium of the original contest.*

4. Contest design

4.1. Optimal prize structure with multiplicatively separable MRTS

In this section, we characterize the optimal prize structure and, in doing so, ask whether strategically equivalent contests call for the same prizes. The contest designer has a total purse of $w > 0$ and wishes to maximize the equilibrium effort. To make the design problem interesting, we assume $n \geq 3$ throughout this section; Section 5 again allows for any $n \geq 2$. We refer to the contest as a *winner-takes-all* contest if $w_1 = w$ and all other prizes are zero, and as a *loser-gets-nothing* contest if $w_1 = \cdots = w_{n-1} = \frac{w}{n-1}$ and $w_n = 0$.

Throughout this section, we focus on contests with a multiplicatively separable MRTS. By Theorem 1 and Corollary 1, such a contest is strategically equivalent to an additive one, which allows us to exploit the results on the additive case from Drugov and Ryvkin (2020b) to derive the optimal prize structure.[15]

In addition, we assume throughout this section that the transformed marginal cost $\frac{c'(e)}{r(e)}$ is strictly increasing in e; equivalently, the transformed cost function $\tilde{c} = c \circ A^{-1}$ is strictly convex. With a multiplicatively separable MRTS, the symmetric first-order condition can be written as $\sum_{j=1}^{n} w_j \int q(x)\, \gamma_j(x)\, dx = \frac{c'(e^*)}{r(e^*)}$, so that, under this assumption, a prize structure maximizes equilibrium effort if and only if it maximizes the players' marginal return to effort, as in Drugov and Ryvkin (2020b).

Recall that we have called two contests with production functions $g_1(e_i, \theta_i)$ and $g_2(e_i, \theta_i)$ strategically equivalent if the equilibria are given by $(e_1^*, \ldots, e_n^*)$ and $(A(e_1^*), \ldots, A(e_n^*))$ when the skill distributions in the two contests are F and $\tilde{F} = F \circ B^{-1}$, and the cost functions c and $\tilde{c} = c \circ A^{-1}$, respectively. Two observations follow.

First, within such a matched pair, the same prize vector maximizes effort in both contests, as the equilibrium efforts are linked by the strictly increasing transformation A. Second, however, this holds for the *transformed* primitives. Holding the skill distribution F fixed, i.e., the situation of a designer who faces given skills under different technologies from the class, the optimal prize structure does vary across strategically equivalent contests. The reason is that the equivalence preserves only ordinal information, i.e., the ranking of outputs, whereas the optimal prize structure depends on the hazard rate of the skill distribution (Drugov and Ryvkin, 2020b), a cardinal feature that the transformation B reshapes. In particular, we will show that the optimal prize structure crucially depends on the monotonicity properties

[15] We maintain the conditions of Drugov and Ryvkin (2020b) for the validity of the first-order approach, imposed on the *transformed* primitives, i.e., the distribution of $B(\Theta_i)$ and the cost function $c \circ A^{-1}$, rather than on F and c.

of the function $h(x) = q\left(F^{-1}(x)\right)\frac{f\left(F^{-1}(x)\right)}{1-x}$. This function depends on the distributional assumptions and the MRTS. It depends on the latter only through q, i.e., the part of the MRTS referring to θ_i. The following proposition characterizes the optimal prize structure under the assumption that h is monotone.

Proposition 2. *i) If h is increasing, then the winner-takes-all contest is optimal.*
ii) If h is decreasing, then the loser-gets-nothing contest is optimal.
iii) If h is constant, any allocation of prizes with $w_n = 0$ is optimal.

As mentioned, the function h depends both on the distributional assumptions and the properties of the production technology. The effects of the distribution on the optimal prize structure are discussed in detail in Drugov and Ryvkin (2020b). To focus on the novel effects of the current analysis, that is, the effects of the production technology, it makes therefore sense to eliminate the influence of the distribution on the prize structure. This is easily accomplished by imposing an exponential distribution with parameter λ such that $F(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$. This distribution has a constant hazard rate, implying that any prize structure with $w_n = 0$ would be optimal in the additive case. For the exponential distribution, we observe $\frac{f\left(F^{-1}(x)\right)}{1-x} = \frac{\lambda \exp\left(-\lambda\left(-\frac{\ln(1-x)}{\lambda}\right)\right)}{1-x} = \lambda$. The function h thus simplifies to $h(x) = \lambda q\left(F^{-1}(x)\right)$. As F^{-1} is monotonically increasing, the monotonicity properties of h only depend on those of q, and the following corollary to Proposition 2 is immediate.

Corollary 2. *Let Θ_i be distributed according to the exponential distribution.*
i) If q is increasing (or $\mathrm{MRTS}(e_i, \theta_i)$ *is increasing in θ_i), then the winner-takes-all contest is optimal.*
ii) If q is decreasing (or $\mathrm{MRTS}(e_i, \theta_i)$ *is decreasing in θ_i), then the loser-gets-nothing contest is optimal.*
iii) If q is constant (or the $\mathrm{MRTS}(e_i, \theta_i)$ *is independent of θ_i), any allocation of prizes with $w_n = 0$ is optimal.*

Corollary 2 shows most clearly that strategically equivalent contests can have different optimal prize structures: under the same skill distribution, contests that are equivalent to each other (and to the additive benchmark, which deems all prize structures with $w_n = 0$

equally good) are optimally designed in opposite ways, with the winner-takes-all and the loser-gets-nothing contest both emerging within a single equivalence class.

The intuition for the corollary is the following. In the symmetric equilibrium, all players exert the same effort. By marginally increasing effort, a player could therefore outperform slightly more skilled opponents. The MRTS measures how effective effort is in overcoming other players' skill advantages and therefore has an impact on effort. If the MRTS is increasing in θ_i, this effect is stronger when a player competes against relatively more able opponents. In a winner-takes-all contest, the "relevant opponent" is the best of the other players (i.e., the largest order statistic) and, thus, relatively able. Accordingly, this type of contest optimally leverages the motivating effect of the increasing MRTS.

In contrast, if the MRTS is decreasing in θ_i, effort is particularly effective in overcoming skill advantages when players compete against relatively unable opponents. This makes the loser-gets-nothing contest optimal, where the "relevant opponent" is the worst of the other players (i.e., the smallest order statistic). Finally, if the MRTS does not depend on θ_i (as in the case of an additive production function), the effectiveness of effort in overcoming other players' skill advantages is independent of the skill of the relevant opponents, implying that all prize structures with $w_n = 0$ are equally good.

As a very simple example, consider a quasilinear production function of the form $g(e_i, \theta_i) = e_i + t(\theta_i)$ (where t is twice continuously differentiable) and continue to impose the exponential distribution. Without having to characterize the equilibrium, we can directly conclude that the winner-takes-all contest is optimal if t is strictly concave, whereas the loser-gets-nothing contest is optimal if t is strictly convex (as $\text{MRTS}(e_i, \theta_i) = 1/t'(\theta_i)$).

The exponential distribution eliminates the distributional force entirely. More generally, the distribution and the production technology interact through a simple product structure: denoting the hazard rate of F by $\eta(\theta) = \frac{f(\theta)}{1-F(\theta)}$, the function h can be written as $h(x) = q(F^{-1}(x))\,\eta(F^{-1}(x))$. As F^{-1} is strictly increasing, the monotonicity of h is that of the product $q \cdot \eta$ on Θ, and Proposition 2 immediately yields the following result.

Corollary 3. *i) If q and η are both increasing, then the winner-takes-all contest is optimal.*
ii) If q and η are both decreasing, then the loser-gets-nothing contest is optimal.
iii) If q and η are differentiable, then the winner-takes-all (loser-gets-nothing) contest is

optimal if $\frac{d}{d\theta} \ln q(\theta) \geq (\leq) - \frac{d}{d\theta} \ln \eta(\theta)$ *for all* $\theta \in \Theta$.

Parts i) and ii) describe situations in which the technology reinforces the distributional force known from the additive case: an increasing (decreasing) hazard rate favors the winner-takes-all (loser-gets-nothing) contest in the additive benchmark (Drugov and Ryvkin, 2020b), and an MRTS that increases (decreases) in skill pushes in the same direction. Part iii) makes precise when the technology overturns the distribution-based recommendation: even if the hazard rate is decreasing, the winner-takes-all contest remains optimal whenever the MRTS increases in skill at a rate exceeding the rate of decline of the hazard rate, and vice versa.

4.2. Optimal prize structure with non-separable MRTS

We conclude this section by considering production functions with non-separable MRTS that lie outside of the equivalence class considered before. The following lemma is helpful in characterizing optimal prize structures.[16]

Lemma 2. *Let* $\frac{\text{MRTS}(e,\theta_i)}{c'(e)}$ *be strictly decreasing in* e *for all* $e \in E$ *and* $\theta_i \in \Theta$. *If there exists a prize structure* $(w_1, ..., w_n)$ *that maximizes*

$$\sum_{j=1}^{n} w_j \int \text{MRTS}(e, x) \gamma_j(x) \, dx$$

for all $e \in E$, *this prize structure maximizes* e^*.

The lemma states that, if there is a prize structure maximizing the players' marginal return to effort, this prize structure is optimal, as it induces maximal effort. While this result is similar to the case of an additive production function, in the current setting, the marginal return to effort depends on the effort level. It is thus conceivable that, depending on the effort level, the marginal return to effort is maximized by different prize structures, in which case the lemma cannot be applied to solve the designer's problem.

[16]Throughout this subsection we maintain that a symmetric pure-strategy equilibrium exists and is characterized by the players' first-order conditions. Concavity of the expected payoff would, as in the additive case, require the second derivative of the cost function to be large relative to that of the expected prize. Because the MRTS now depends on effort, this is more complex than in the additive case.

That being said, in many instances, the same prize structure maximizes the marginal return to exerting effort for all effort levels. As we show in the upcoming proposition, the effects of the MRTS on the optimal prize structure are then similar to those from the preceding subsection with a multiplicatively separable MRTS. Define $\hat{h}(e,x) = \text{MRTS}\left(e, F^{-1}(x)\right) \frac{f\left(F^{-1}(x)\right)}{1-x}$. We show the following.[17]

Proposition 3. *Let $\frac{\text{MRTS}(e,\theta_i)}{c'(e)}$ be strictly decreasing in e for all $e \in E$ and $\theta_i \in \Theta$, as in Lemma 2.*
i) If $\hat{h}(e,x)$ is increasing in x for all e, then the winner-takes-all contest is optimal.
ii) If $\hat{h}(e,x)$ is decreasing in x for all e, then the loser-gets-nothing contest is optimal.

5. Noise and effort in the additive equivalence class

Our characterization also allows existing comparative-static results for additive contests to be extended to the entire equivalence class. As a second application, consider the analysis of noise by Drugov and Ryvkin (2020a). They show that, in additive contests, one noise distribution generates lower equilibrium effort than another for every admissible prize structure and every number of contestants if and only if it is more dispersed in the dispersive order. For winner-takes-all contests, they further characterize equilibrium effort using the Rényi entropy of a suitable order statistic.

Suppose throughout this section that $\text{MRTS}(e_i, \theta_i) = r(e_i)q(\theta_i)$ where $r(e_i), q(\theta_i) > 0$. By Corollary 1, there exist strictly increasing transformations

$$A(e_i) = \int^{e_i} r(s)\, ds \quad \text{and} \quad B(\theta_i) = \int^{\theta_i} \frac{1}{q(s)}\, ds$$

such that the contest is strategically equivalent to one with additive production function $\tilde{g}(\tilde{e}_i, \tilde{\theta}_i) = \tilde{e}_i + \tilde{\theta}_i$, where $\tilde{e}_i = A(e_i)$ and $\tilde{\theta}_i = B(\theta_i)$, and the transformed cost function is $\tilde{c}(\tilde{e}_i) = c\big(A^{-1}(\tilde{e}_i)\big)$.

We assume that the transformed skill variable $B(\Theta_i)$ is admissible in the sense of Section 2. Throughout, the dispersive order is used in the strict sense of Drugov and Ryvkin (2020a, Definition 1). Hence, every result in Drugov and Ryvkin (2020a) immediately applies to the

[17] Notice that a non-separable MRTS implies that $\hat{h}(e,x)$ cannot be constant in x.

transformed skill variable $B(\Theta_i)$. Since A is strictly increasing, any ranking of transformed equilibrium efforts is also a ranking of equilibrium efforts in the original contest.

Let F and f denote the distribution and density of Θ_i, and let $\tilde{F}$ and $\tilde{f}$ denote those of $B(\Theta_i)$. Since $\tilde{F}(B(\theta_i)) = F(\theta_i)$ and $B'(\theta_i) = 1/q(\theta_i)$, we have $\tilde{f}(B(\theta_i)) = q(\theta_i)f(\theta_i)$.

It follows that the inverse quantile density of the transformed skill distribution is

$$\tilde{m}(z) = \tilde{f}\big(\tilde{F}^{-1}(z)\big) = q\big(F^{-1}(z)\big)f\big(F^{-1}(z)\big).$$

Thus, the inverse quantile density $f(F^{-1}(z))$ from the additive model is replaced by the technology-adjusted inverse quantile density $m_F^q(z) := q\big(F^{-1}(z)\big)f\big(F^{-1}(z)\big)$.

The following result is an immediate implication of Proposition 1 in Drugov and Ryvkin (2020a).

Corollary 4. *Consider two skill distributions F_X and F_Y under the same production technology g. Suppose that the symmetric equilibrium exists and is characterized by the first-order condition, and that the transformed marginal cost $\frac{c'(e_i)}{r(e_i)}$ is strictly increasing. Then $e_X^* < e_Y^*$ for every admissible prize schedule and every number of contestants n if and only if $B(X)$ is more dispersed than $B(Y)$ in the dispersive order.*

Corollary 4 shows that the relevant notion of dispersion is no longer that of the original skill distribution but of the transformed skill variable $B(\Theta_i)$. Equivalently, the production technology modifies the inverse quantile density by the factor $q(F^{-1}(z))$.

The remaining results in Drugov and Ryvkin (2020a) extend in exactly the same way. In particular, their entropy characterization for winner-takes-all contests applies directly to the transformed skill variable $B(\Theta_i)$. Likewise, all quantile stochastic dominance results continue to hold after replacing the inverse quantile density $f(F^{-1}(z))$ by $q(F^{-1}(z))f(F^{-1}(z))$.

Thus, the entire comparative-static analysis of noise developed for additive contests carries over to the complete equivalence class characterized in Section 3.

The two applications developed above are illustrative rather than exhaustive. The device is always the same: a result derived for the additive Lazear–Rosen tournament applies to any contest in $[g_{add}^c]$ once its hypotheses are imposed on the transformed primitives, that is, on the distribution of $B(\Theta_i)$ and the cost function $c \circ A^{-1}$ rather than on F and c. Therefore, such a transfer is easy because no argument has to be redone, but results derived in the additive literature need not extend to the transformed environment.

6. Conclusion

This paper characterizes the production technologies that generate contests strategically equivalent to the standard additive Lazear–Rosen tournament. While the contest literature has established strategic equivalence for particular production technologies, a general characterization has been missing. We show that such an equivalence exists if and only if the MRTS between effort and skill is multiplicatively separable. This condition therefore provides a complete characterization of the production technologies that belong to the equivalence class of the additive benchmark. The equivalence operates at the level of the normal-form game, so that all pure-strategy equilibria correspond, also with heterogeneous or risk-averse players, and the underlying ordinal condition is sufficient and, within the symmetric framework, also necessary for first-order behavior to correspond across all environments.

The characterization has two main implications. First, it determines the exact scope of the additive model: the equivalence class is large, containing the multiplicative, Cobb–Douglas, CES, and quasilinear technologies, and equilibrium behavior in any of these environments can be analyzed by transforming the contest into the additive framework, so that existing results derived for additive contests apply. Second, strategic equivalence does not extend to contest design: the equivalence preserves only ordinal information, whereas optimal prize structures depend on cardinal features of the environment, so that, holding the skill distribution fixed, strategically equivalent technologies can call for opposite prize structures. Methodologically, the characterization links the analysis of strategically equivalent contests to the classical literature on additive utility representations, showing that both problems share the same underlying mathematical structure.

We illustrate the usefulness of the characterization in two applications. First, building on existing results for additive contests, we derive optimal prize structures for the entire equivalence class. In particular, we show that when all prize structures are equivalent under the additive benchmark, the monotonicity of the MRTS determines the optimal design: an MRTS that increases in skill favors winner-takes-all contests, whereas an MRTS that decreases in skill favors loser-gets-nothing contests, so that opposite designs emerge within the equivalence class. More generally, the optimal design depends on the monotonicity of the product of the skill component of the MRTS and the hazard rate of the skill distribution.

Second, we show that the comparative-static analysis of noise developed by Drugov and Ryvkin (2020a) also extends immediately to all strategically equivalent production technologies. The relevant stochastic-order and entropy characterizations therefore continue to hold after transforming the skill variable according to the production technology. More generally, any result derived for the additive tournament transfers in the same way, provided that it concerns the ranking of individual equilibrium efforts rather than cardinal magnitudes such as aggregate effort.

An important direction for future research is the analysis of contests whose production technologies fall outside the characterized equivalence class. While we provide sufficient conditions under which standard prize structures remain optimal in some of these environments, a more complete understanding of contest design with non-separable MRTS remains an interesting avenue for future research.

Appendix A. Proofs of the main results

Proof of Lemma 1. Regarding reflexivity, we observe that $g^c \sim g^c$ by setting $\phi = A = B = id$. Regarding symmetry, suppose that $g_1^c \sim g_2^c$. As ϕ, A, and B are increasing, the inverse functions ϕ^{-1}, A^{-1}, and B^{-1} exist. If we define $\tilde{e}_i := A(e_i) \Leftrightarrow e_i = A^{-1}(\tilde{e}_i)$ and $\tilde{\theta}_i := B(\theta_i) \Leftrightarrow \theta_i = B^{-1}(\tilde{\theta}_i)$, we obtain $\phi^{-1}\left(g_2\left(\tilde{e}_i, \tilde{\theta}_i\right)\right) = g_1\left(A^{-1}(\tilde{e}_i), B^{-1}(\tilde{\theta}_i)\right)$, implying that $g_2^c \sim g_1^c$.

Finally, suppose that $g_1^c \sim g_2^c$ and $g_2^c \sim g_3^c$ for three contests. Denote the three functions in the equivalence of g_1^c and g_2^c by ϕ_1 A_1, and B_1, and those in the equivalence of g_2^c and g_3^c by ϕ_2, A_2, and B_2. Further, denote the variables in the first equivalence by e_i and θ_i and in the second by $\tilde{e}_i$ and $\tilde{\theta}_i$. We observe that $\phi_1\left(g_1\left(e_i, \theta_i\right)\right) = g_2\left(A_1\left(e_i\right), B_1\left(\theta_i\right)\right)$ and $\phi_2\left(g_2\left(\tilde{e}_i, \tilde{\theta}_i\right)\right) = g_3\left(A_2\left(\tilde{e}_i\right), B_2\left(\tilde{\theta}_i\right)\right)$. Defining $\tilde{e}_i := A_1(e_i)$ and $\tilde{\theta}_i := B_1(\theta_i)$, the second equality can be written as $\phi_2\left(g_2\left(A_1(e_i), B_1(\theta_i)\right)\right) = g_3\left(A_2\left(A_1(e_i)\right), B_2\left(B_1(\theta_i)\right)\right)$. It follows that

$$\phi_2\left(\phi_1\left(g_1\left(e_i, \theta_i\right)\right)\right) = \phi_2\left(g_2\left(A_1(e_i), B_1(\theta_i)\right)\right) = g_3\left(A_2\left(A_1(e_i)\right), B_2\left(B_1(\theta_i)\right)\right),$$

from which we conclude that $g_1^c \sim g_3^c$ so that transitivity is fulfilled as well. □

Proof of Theorem 1. Applying the results of Samuelson (1947, p.176) to our problem, a production function can be transformed into an additively separable function if and only if $\frac{\partial^2}{\partial e_i \partial \theta_i} \ln \mathrm{MRTS}\left(e_i, \theta_i\right) = 0$. It remains to show that this condition is equivalent to the MRTS being multiplicatively separable.

Let the MRTS be multiplicatively separable, i.e., $\mathrm{MRTS}\left(e_i, \theta_i\right) = r\left(e_i\right) q\left(\theta_i\right)$. We observe

$$\begin{aligned} & \ln \mathrm{MRTS}\left(e_i, \theta_i\right) = \ln r\left(e_i\right) + \ln q\left(\theta_i\right) \\ \Rightarrow \quad & \frac{\partial^2}{\partial e_i \partial \theta_i} \ln \mathrm{MRTS}\left(e_i, \theta_i\right) = 0. \end{aligned}$$

Now let $\frac{\partial^2}{\partial e_i \partial \theta_i} \ln \mathrm{MRTS}\left(e_i, \theta_i\right) = 0$. Then, $\ln \mathrm{MRTS}\left(e_i, \theta_i\right)$ can be written as $\ln \mathrm{MRTS}\left(e_i, \theta_i\right) = \hat{r}\left(e_i\right) + \hat{q}\left(\theta_i\right)$ for functions $\hat{r}$ and $\hat{q}$ of one variable each (see Samuelson, 1947, p.176, or Theorem 3.2.3 in Boyd, 2023). It follows that

$$\begin{aligned} \mathrm{MRTS}\left(e_i, \theta_i\right) &= \exp\left(\hat{r}\left(e_i\right) + \hat{q}\left(\theta_i\right)\right) \\ &= \exp\left(\hat{r}\left(e_i\right)\right) \exp\left(\hat{q}\left(\theta_i\right)\right), \end{aligned}$$

meaning that the MRTS (e_i, θ_i) is multiplicatively separable.

$\square$

Proof of Corollary 1. By Theorem 1, there exist strictly increasing functions ϕ, $\hat{A}$, and $\hat{B}$ with $\phi(g(e_i, \theta_i)) = \hat{A}(e_i) + \hat{B}(\theta_i)$ for all $e_i \in E$ and $\theta_i \in \Theta$. Moreover, given our smoothness assumptions, the additive representation obtained via the construction in Samuelson (1947, p.175–176) is differentiable. We can thus calculate the corresponding MRTS as

$$\mathrm{MRTS}(e_i, \theta_i) = \frac{\hat{A}'(e_i)}{\hat{B}'(\theta_i)}.$$

Together with $\mathrm{MRTS}(e_i, \theta_i) = r(e_i)q(\theta_i)$, we obtain $\frac{\hat{A}'(e_i)}{r(e_i)} = q(\theta_i)\hat{B}'(\theta_i)$ for all e_i and θ_i. As the left-hand side does not depend on θ_i and the right-hand side does not depend on e_i, both sides equal a constant k, which is strictly positive because $\hat{A}' > 0$ and $r > 0$. Hence $\hat{A}'(e_i) = k\,r(e_i)$ and $\hat{B}'(\theta_i) = k/q(\theta_i)$, so that $\hat{A}$ and $\hat{B}$ coincide with A and B from the corollary's statement up to the common scaling factor k and additive constants. Absorbing these into ϕ delivers a strictly increasing ϕ with $\phi(g(e_i, \theta_i)) = A(e_i) + B(\theta_i)$. $\square$

Proof of Proposition 1. With $n = 2$, we have $\gamma_1(x) = f(x)^2$ and $\gamma_2(x) = -f(x)^2$, so that the symmetric first-order condition of contest 1 under (F, c) can be stated as

$$(w_1 - w_2) \int \mathrm{MRTS}_1(e, x)\, f(x)^2\, dx = c'(e).$$

Consider the symmetric first-order condition of contest 2 under $(\tilde{F}, \tilde{c}) = (F \circ B^{-1}, c \circ A^{-1})$, evaluated at $\tilde{e} = A(e)$:

$$(w_1 - w_2) \int \mathrm{MRTS}_2(A(e), x)\, \tilde{f}(x)^2\, dx = \tilde{c}'(A(e)).$$

Substituting $x = B(y)$, using $\tilde{f}(B(y)) = f(y)/B'(y)$, $dx = B'(y)\, dy$, and $\tilde{c}'(A(e)) = c'(e)/A'(e)$, and multiplying by $A'(e)$, this condition becomes

$$(w_1 - w_2) \int \hat{M}(e, y)\, f(y)^2\, dy = c'(e), \qquad \text{where } \hat{M}(e, y) := \mathrm{MRTS}_2\left(A(e), B(y)\right) \frac{A'(e)}{B'(y)}.$$

Define $D(e, y) := \hat{M}(e, y) - \mathrm{MRTS}_1(e, y)$, which is a continuous function on $E \times \Theta$.

The remainder of the proof proceeds in two steps. In Step 1, we show that $D \equiv 0$ on $E \times \Theta$. Fix an arbitrary e_0 in the interior of E and an arbitrary admissible F, and define $v := (w_1 - w_2)\int \mathrm{MRTS}_1(e_0, x)f(x)^2\,dx > 0$. The cost function defined by $c'(e) = v\exp\left(K(e-e_0)\right)$, $K > 0$, with its level chosen such that $c(\underline{e}) \leq w_n$, is admissible, and e_0 solves the first-order condition of contest 1 by construction.

By hypothesis, $A(e_0)$ then solves the first-order condition of contest 2 under $(\tilde{F}, \tilde{c})$, and subtracting the two conditions yields

$$\int D(e_0, y)\, f(y)^2\, dy = 0 \qquad \text{for every admissible } F.$$

Now fix an arbitrary interior point $y_0 \in \Theta$, let u_h denote the uniform density on $(y_0 - h/2,\ y_0 + h/2)$ (where $0 < h < 1$ is so small that this support is contained in Θ), let $\bar{f}$ be an arbitrary bounded admissible density with support Θ, and set $f_h := (1-h)u_h + h\bar{f}$, which has support Θ and is admissible. Then

$$h\int D(e_0, y) f_h(y)^2\, dy = h\int D\left(e_0, y\right)\left(\left(1-h\right)^2 u_h\left(y\right)^2 + h^2 \bar{f}\left(y\right)^2 + 2\left(1-h\right) u_h\left(y\right) h \bar{f}\left(y\right)\right) dy.$$

This can be rewritten as

$$\left(1-h\right)^2 \frac{1}{h}\int_{y_0-\frac{h}{2}}^{y_0+\frac{h}{2}} D\left(e_0, y\right) dy + h^3 \int D\left(e_0, y\right) \bar{f}\left(y\right)^2 dy + 2\left(1-h\right) h \int_{y_0-\frac{h}{2}}^{y_0+\frac{h}{2}} D\left(e_0, y\right) \bar{f}\left(y\right) dy,$$

or, more compactly, as

$$(1-h)^2 \frac{1}{h}\int_{y_0-h/2}^{y_0+h/2} D(e_0, y)\, dy + O(h).$$

By the continuity of D, it follows that this last expression approaches $D(e_0, y_0)$ as $h \downarrow 0$. As $h\int D(e_0, y) f_h(y)^2\, dy = 0$ for every h, we conclude $D(e_0, y_0) = 0$. By continuity, the same equality holds for boundary points, thus, $D \equiv 0$ on $E \times \Theta$.

Next, we proceed to Step 2, where we construct ϕ. Define $\hat{g}(e, \theta) := g_2(A(e), B(\theta))$ on $E \times \Theta$ and note that its MRTS equals $\hat{M}$. By Step 1, $\hat{g}$ and g_1 have identical MRTS on $E \times \Theta$. The isoquants of either function are the solution curves of the ordinary differential equation $\frac{d\theta}{de} = -\,\mathrm{MRTS}_1(e, \theta)$. Since MRTS_1 is continuously differentiable, the solution through any point of $E \times \Theta$ is unique, so the two families of isoquants coincide. Moreover, every level set of g_1 is connected: since g_1 is strictly increasing in both arguments, the set of efforts e for which the level set $\{g_1 = v\}$ contains a point (e, θ) with $\theta \in \Theta$ is an interval, on which the level set is the graph of a continuous, strictly decreasing function $\theta = \tau_v(e)$. We now define

$$\phi(v) := \hat{g}(e, \theta) \qquad \text{for an arbitrary } (e, \theta) \text{ with } g_1(e, \theta) = v.$$

This function is well defined, as $\hat{g}$ is constant along each isoquant of g_1. To see that ϕ is strictly increasing, take $v < v'$ in the range of g_1. If some e lies in the e-intervals of both level sets, then $\tau_v(e) < \tau_{v'}(e)$ and thus $\phi(v) = \hat{g}(e, \tau_v(e)) < \hat{g}(e, \tau_{v'}(e)) = \phi(v')$, as $\hat{g}$ is strictly increasing in its second argument. The general case (where the two level sets need not overlap in their e-intervals) follows by chaining this comparison through intermediate values, the range of g_1 being an interval. By construction, $\phi(g_1(e, \theta)) = \hat{g}(e, \theta) = g_2(A(e), B(\theta))$ for all $(e, \theta) \in E \times \Theta$, i.e., $g_1^c \sim g_2^c$. □

Proof of Theorem 2. For ease of notation, we write the production function as $g\left(y_1, ..., y_M\right)$ in this proof, where $M = m + 1 \geq 3$ denotes the total number of variables (the m efforts and the skill). Define $\text{MRTS}_{ij} = \frac{\partial g / \partial y_i}{\partial g / \partial y_j}$ for $i \neq j$. Applying Theorem 1 in Goldman and Uzawa (1964) to the singleton partition, $g\left(y_1, ..., y_M\right)$ can be written as $g\left(y_1, ..., y_M\right) = \psi\left(A^1\left(y_1\right) + ... + A^M\left(y_M\right)\right)$ (with ψ and all A^k strictly increasing) if and only if $\frac{\partial(\text{MRTS}_{ij})}{\partial y_k} = 0$ for all i, j, k with $i \neq j$, $i \neq k$, and $j \neq k$. We perform the proof by showing that the latter condition is equivalent to $\text{MRTS}_{ij} = q_i\left(y_i\right) r_j\left(y_j\right)$ for all i and j with $i \neq j$, where the functions q_i and r_j may differ across variables but do not depend on any of the other variables.

We prove both directions. Let $\text{MRTS}_{ij} = q_i\left(y_i\right) r_j\left(y_j\right)$ for all i and j with $i \neq j$. As q_i and r_j are independent of y_k, it immediately follows that $\frac{\partial(\text{MRTS}_{ij})}{\partial y_k} = 0$.

Next, let $\frac{\partial(\text{MRTS}_{ij})}{\partial y_k} = 0$ for all i, j, k with $i \neq j$, $i \neq k$, and $j \neq k$. Notice that

$$\text{MRTS}_{ij} = \frac{\text{MRTS}_{ik}}{\text{MRTS}_{jk}}.$$

Taking logarithms, define

$$m_{ij} = \log \text{MRTS}_{ij}\,.$$

Then

$$m_{ij}(y_i, y_j) = m_{ik}(y_i, y_k) - m_{jk}(y_j, y_k).$$

Since the left-hand side is independent of y_k, differentiation with respect to y_k yields

$$0 = \frac{\partial m_{ik}(y_i, y_k)}{\partial y_k} - \frac{\partial m_{jk}(y_j, y_k)}{\partial y_k},$$

or equivalently,

$$\frac{\partial m_{ik}(y_i, y_k)}{\partial y_k} = \frac{\partial m_{jk}(y_j, y_k)}{\partial y_k}.$$

The left-hand side depends only on (y_i, y_k), while the right-hand side depends only on (y_j, y_k). Since the equality holds for all (y_i, y_j, y_k), both sides must in fact depend only on y_k. Hence there exists a function ω_k such that

$$\frac{\partial m_{ik}(y_i, y_k)}{\partial y_k} = \omega_k(y_k)$$

for every $i \neq k$. Integrating with respect to y_k, $m_{ik}(y_i, y_k) = \lambda_{ik}(y_i) + \Omega_k(y_k)$, where λ_{ik} and Ω_k are functions of one variable. Exponentiating, we obtain $\mathrm{MRTS}_{ik}(y_i, y_k) = \exp(\lambda_{ik}(y_i)) \exp(\Omega_k(y_k))$ from which we conclude that the MRTS_{ik} is multiplicatively separable, with the respective functions being independent of all the other variables.

It remains to check that the factors can be chosen to depend only on their own index. Fix a reference index k and define, for $i \neq k$, $q_i(y_i) := \exp(\lambda_{ik}(y_i))$ and $r_i(y_i) := \exp(-\lambda_{ik}(y_i))$, together with $q_k(y_k) := \exp(-\Omega_k(y_k))$ and $r_k(y_k) := \exp(\Omega_k(y_k))$. Using the identity $\mathrm{MRTS}_{ij} = \mathrm{MRTS}_{ik} \,/\, \mathrm{MRTS}_{jk}$ and $m_{ik}(y_i, y_k) = \lambda_{ik}(y_i) + \Omega_k(y_k)$, we obtain, for all $i \neq j$,

$$\mathrm{MRTS}_{ij}(y_i, y_j) = \exp\big(m_{ik} - m_{jk}\big) = \exp(\lambda_{ik}(y_i)) \exp(-\lambda_{jk}(y_j)) = q_i(y_i)\, r_j(y_j).$$

For the case $j = k$, substituting the definitions into $\mathrm{MRTS}_{ik}(y_i, y_k) = \exp(\lambda_{ik}(y_i)) \exp(\Omega_k(y_k))$ gives $\mathrm{MRTS}_{ik} = q_i(y_i) r_k(y_k)$. For the case $i = k$, we similarly obtain $\mathrm{MRTS}_{kj} = \frac{1}{\mathrm{MRTS}_{jk}} = \exp(-\lambda_{jk}(y_j)) \exp(-\Omega_k(y_k)) = r_j(y_j) q_k(y_k)$. Each factor depends only on its own variable, completing the proof. $\square$

Proof of Proposition 2. As described in the text, the proof exploits the equivalence of a contest with a general production function and an additive one and uses the results on the additive case from Drugov and Ryvkin (2020b).

Recall that transformation of a contest with a general production function $g(e_i, \theta_i)$ into the additive case requires the existence of monotonically increasing functions ϕ, A, and B such that $\phi(g(e_i, \theta_i)) = A(e_i) + B(\theta_i)$. The production function in the additive case would be given by the expression on the right-hand-side of the equation. And according to Drugov and Ryvkin (2020b), the optimal prize structure would depend on the distribution of the random variable $B(\Theta_i)$, and in particular, its hazard rate.

We observe

$$B(\theta_i) \leq t \Leftrightarrow \theta_i \leq B^{-1}(t),$$

such that $\tilde{F}(t) = F(B^{-1}(t))$. Accordingly,

$$\tilde{f}(t) = \frac{f(B^{-1}(t))}{B'(B^{-1}(t))}.$$

It follows that the hazard rate of the transformed skill variable can be stated as

$$\begin{aligned}\tilde{\eta}(t) &= \frac{\frac{f\left(B^{-1}(t)\right)}{B'(B^{-1}(t))}}{1 - F(B^{-1}(t))} \\ &= \frac{1}{B'(B^{-1}(t))}\frac{f(B^{-1}(t))}{1 - F(B^{-1}(t))}.\end{aligned}$$

With a multiplicatively separable MRTS of the form $\text{MRTS}(e_i, \theta_i) = r(e_i)\, q(\theta_i)$, we know from Corollary 1 that $B'(\theta_i) = \frac{1}{q(\theta_i)}$ (up to an irrelevant positive scaling factor), implying that $\tilde{\eta}$ can be stated as

$$\tilde{\eta}(t) = q\left(B^{-1}(t)\right)\frac{f(B^{-1}(t))}{1 - F(B^{-1}(t))}.$$

As shown in Drugov and Ryvkin (2020b), the argument in the hazard rate is replaced by $t = \tilde{F}^{-1}(x)$, so that the expression becomes

$$\tilde{\eta}\left(\tilde{F}^{-1}(x)\right) = q\left(B^{-1}\left(\tilde{F}^{-1}(x)\right)\right)\frac{f\left(B^{-1}\left(\tilde{F}^{-1}(x)\right)\right)}{1 - F\left(B^{-1}\left(\tilde{F}^{-1}(x)\right)\right)}.$$

From $\tilde{F}(t) = F(B^{-1}(t))$, it follows that

$$F\left(B^{-1}(t)\right) = x \Leftrightarrow t = B\left(F^{-1}(x)\right),$$

implying that $\tilde{F}^{-1}(x) = B(F^{-1}(x))$. The function $\tilde{\eta}$ can be simplified to

$$\begin{aligned}\tilde{\eta}\left(\tilde{F}^{-1}(x)\right) &= q\left(B^{-1}\left(B\left(F^{-1}(x)\right)\right)\right)\frac{f(B^{-1}(B(F^{-1}(x))))}{1 - F(B^{-1}(B(F^{-1}(x))))} \\ &= q\left(F^{-1}(x)\right)\frac{f(F^{-1}(x))}{1 - x} =: h(x).\end{aligned}$$

The statements in the proposition then follow immediately from Proposition 2 in Drugov and Ryvkin (2020b). □

Proof of Corollary 3. As shown in the proof of Proposition 2, $h(x) = q(F^{-1}(x))\frac{f\left(F^{-1}(x)\right)}{1-x}$. Since $1 - x = 1 - F(F^{-1}(x))$, this can be written as $h(x) = q(F^{-1}(x))\,\eta(F^{-1}(x))$. As F^{-1} is strictly increasing, h is increasing (decreasing) if and only if the product $q \cdot \eta$ is increasing (decreasing) on Θ. Parts i) and ii) then follow from Proposition 2. For part iii), note that, with $q, \eta > 0$ differentiable, $q\eta$ is increasing (decreasing) if $(\ln(q\eta))' = (\ln q)' + (\ln \eta)' \geq (\leq)\ 0$ on Θ. $\square$

Proof of Lemma 2. We first show that, for every admissible prize structure, $\Phi(x) := \sum_{j=1}^{n} w_j \gamma_j(x) \geq 0$ for all $x \in \Theta$. By Abel summation,

$$\sum_{j=1}^{n} w_j \gamma_j(x) = \sum_{k=1}^{n-1} (w_k - w_{k+1}) S_k(x) + w_n S_n(x), \qquad \text{where } S_k(x) := \sum_{j=1}^{k} \gamma_j(x).$$

The computation in the proof of Proposition 3 establishes the identity

$$S_k(x) = k \binom{n-1}{k} F(x)^{n-k-1} (1 - F(x))^{k-1} f(x)^2 \geq 0,$$

which also yields $S_n(x) = 0$ (as $\binom{n-1}{n} = 0$). Since $w_k \geq w_{k+1}$ for all k, we conclude that $\Phi(x) \geq 0$; moreover, since at least one prize inequality is strict and $0 < F(x) < 1$ as well as $f(x) > 0$ in the interior of Θ, Φ is strictly positive on a set of positive measure.

The first-order condition characterizing equilibrium effort can be written as

$$G(e^*) := \frac{1}{c'(e^*)} \sum_{j=1}^{n} w_j \int \mathrm{MRTS}(e^*, x) \gamma_j(x)\, dx = \int \frac{\mathrm{MRTS}(e^*, x)}{c'(e^*)} \Phi(x)\, dx = 1.$$

As $\frac{\mathrm{MRTS}(e,x)}{c'(e)}$ is strictly decreasing in e for all x and $\Phi \geq 0$ pointwise (and positive on a set of positive measure), G is strictly decreasing in e.

Now let $(w_1, ..., w_n)$ be a prize structure that maximizes $\sum_{j=1}^{n} w_j \int \mathrm{MRTS}(e, x) \gamma_j(x)\, dx$ for all $e \in E$, with associated equilibrium effort e^*, and let $(\hat{w}_1, ..., \hat{w}_n)$ be an arbitrary admissible prize structure with associated equilibrium effort $\hat{e}^*$. Denoting by $\hat{G}$ the counterpart of G for $(\hat{w}_1, ..., \hat{w}_n)$, we obtain

$$G(\hat{e}^*) \geq \hat{G}(\hat{e}^*) = 1 = G(e^*).$$

As G is strictly decreasing in e, it follows that $e^* \geq \hat{e}^*$, i.e., $(w_1, ..., w_n)$ maximizes equilibrium effort. $\square$

Proof of Proposition 3. The proof closely follows that of Drugov and Ryvkin (2020b).

We define $\hat{\beta}_j(e) = \int \mathrm{MRTS}\,(e,x)\,\gamma_j\,(x)\,dx$. Referring to Lemma 2, for a given effort e, we search for a prize structure that solves the following maximization problem:

$$\max_{w_1,\ldots,w_n} \sum_{j=1}^{n} \hat{\beta}_j(e) w_j \qquad \text{s.t. } w_1 \geq w_2 \geq \ldots \geq w_n \geq 0, \quad \sum_{j=1}^{n} w_j = w.$$

As $\gamma_n\,(x) = -\,(n-1)\,(1-F\,(x))^{n-2}\,(f\,(x))^2 < 0$, it follows that $\hat{\beta}_n(e) < 0$ so that $w_n = 0$ is optimal. Following Drugov and Ryvkin (2020b), the optimization problem can be restated as

$$\max_{d_1,\ldots,d_{n-1}} \sum_{j=1}^{n-1} B_j(e) d_j \qquad \text{s.t. } d_1, d_2, \ldots, d_{n-1} \geq 0, \quad \sum_{j=1}^{n-1} j d_j = w,$$

with $d_j = w_j - w_{j+1}$ and $B_j(e) = \sum_{k=1}^{j} \hat{\beta}_k(e)$.

To understand this, notice first that the monotonicity constraint obviously transforms into $d_1, d_2, \ldots, d_{n-1} \geq 0$. Further, observe that

$$\begin{aligned}\sum_{j=1}^{n-1} j d_j &= (w_1 - w_2) + 2\,(w_2 - w_3) + \ldots + (n-1)\,(w_{n-1} - w_n) \\ &= w_1 + w_2 + w_3 + \ldots + w_{n-1} - (n-1)\,w_n = \sum_{j=1}^{n} w_j,\end{aligned}$$

as $w_n = 0$. Finally, notice that

$$\begin{aligned}\sum_{j=1}^{n-1} B_j(e) d_j &= \sum_{j=1}^{n-1} B_j(e)\,(w_j - w_{j+1}) = \sum_{j=1}^{n-1} B_j(e) w_j - \sum_{j=1}^{n-1} B_j(e) w_{j+1} \\ &= \sum_{j=1}^{n-1} B_j(e) w_j - \sum_{j=2}^{n} B_{j-1}(e) w_j = B_1(e) w_1 + \sum_{j=2}^{n-1} \left(B_j(e) - B_{j-1}(e)\right) w_j - B_{n-1}(e) w_n.\end{aligned}$$

As $B_1(e) = \hat{\beta}_1(e)$, $B_j(e) - B_{j-1}(e) = \hat{\beta}_j(e)$ and $w_n = 0$, this simplifies to

$$\hat{\beta}_1(e) w_1 + \sum_{j=2}^{n-1} \hat{\beta}_j(e) w_j = \sum_{j=1}^{n} \hat{\beta}_j(e) w_j.$$

The maximization problem

$$\max_{d_1,\ldots,d_{n-1}} \sum_{j=1}^{n-1} B_j(e) d_j \qquad \text{s.t. } d_1, d_2, \ldots, d_{n-1} \geq 0, \quad \sum_{j=1}^{n-1} j d_j = w$$

corresponds to a utility maximization problem with perfect substitutes. It follows that there is an optimal solution, in which exactly one of the d_l is positive, while all other are equal to zero. The positive one is the one maximizing

$$\begin{aligned}\frac{B_j(e)}{j} &= \frac{1}{j}\left(\sum_{k=1}^{j}\hat{\beta}_k(e)\right)\\ &= \frac{1}{j}\left(\sum_{k=1}^{j}\int \mathrm{MRTS}\left(e,x\right)\gamma_k\left(x\right)dx\right).\end{aligned}$$

Following Drugov and Ryvkin (2020b), this can be stated as

$$\frac{B_j(e)}{j} = \frac{1}{n}\int_0^1 \mathrm{MRTS}\left(e,F^{-1}\left(x\right)\right)\frac{f\left(F^{-1}\left(x\right)\right)}{1-x}dF^B\left(x,n-j,j+1\right),$$

where $F^B\left(x,n-j,j+1\right)$ is the cdf of the beta distribution with parameters $(n-j,j+1)$. To understand this, notice that

$$\begin{aligned}\frac{B_j(e)}{j} &= \frac{1}{j}\left(\sum_{k=1}^{j}\int \mathrm{MRTS}(e,x)\gamma_k(x)dx\right)\\ &= \frac{1}{j}\left(\sum_{k=1}^{j}\int \mathrm{MRTS}(e,x)\binom{n-1}{k-1}F(x)^{n-k-1}(1-F(x))^{k-2}(n-k-(n-1)F(x))f(x)dF(x)\right).\end{aligned}$$

When using the change of variable, $z=F(x)$, this becomes

$$\begin{aligned}\frac{B_j(e)}{j} &= \frac{1}{j}\left(\sum_{k=1}^{j}\int_0^1 \mathrm{MRTS}(e,F^{-1}(z))\frac{f(F^{-1}(z))}{1-z}\binom{n-1}{k-1}z^{n-k-1}(1-z)^{k-1}(n-k-(n-1)z)dz\right)\\ &= \frac{1}{j}\left(\int \mathrm{MRTS}(e,F^{-1}(z))\frac{f(F^{-1}(z))}{1-z}\sum_{k=1}^{j}\left(\binom{n-1}{k-1}z^{n-k-1}(1-z)^{k-1}(n-k-(n-1)z)\right)dz\right).\end{aligned}$$

Now notice that

$$
\begin{aligned}
& \sum_{k=1}^{j}\left(\binom{n-1}{k-1} z^{n-k-1}(1-z)^{k-1}(n-k-(n-1) z)\right) \\
= & \sum_{k=1}^{j}\left(\binom{n-1}{k-1} z^{n-k-1}(1-z)^{k-1}((n-k)(1-z)-(k-1) z)\right) \\
= & \sum_{k=1}^{j}\left((n-k)\binom{n-1}{k-1} z^{n-(k+1)}(1-z)^{k}-(k-1)\binom{n-1}{k-1} z^{n-k}(1-z)^{k-1}\right) \\
= & \sum_{k=1}^{j}\left(k\binom{n-1}{k} z^{n-(k+1)}(1-z)^{k}-(k-1)\binom{n-1}{k-1} z^{n-k}(1-z)^{k-1}\right) \\
= & \sum_{k=1}^{j}\left(k\binom{n-1}{k} z^{n-(k+1)}(1-z)^{k}\right)-\sum_{k=0}^{j-1}\left(k\binom{n-1}{k} z^{n-(k+1)}(1-z)^{k}\right) \\
= & j\binom{n-1}{j} z^{n-(j+1)}(1-z)^{j} .
\end{aligned}
$$

The last three equalities use, in turn, the identity $(n-k)\binom{n-1}{k-1}=k\binom{n-1}{k}$, a reindexing of the second sum ($k \rightarrow k+1$), and the telescoping of the two resulting sums, which leaves only the term $k=j$ of the first sum (the $k=0$ term of the second sum vanishes). It follows that $\frac{B_j(e)}{j}$ simplifies to

$$
\begin{aligned}
\frac{B_{j}(e)}{j} & =\left(\int \operatorname{MRTS}\left(e, F^{-1}(z)\right) \frac{f\left(F^{-1}(z)\right)}{1-z}\binom{n-1}{j} z^{n-(j+1)}(1-z)^{j} d z\right) \\
& =\frac{1}{n}\left(\int \operatorname{MRTS}\left(e, F^{-1}(z)\right) \frac{f\left(F^{-1}(z)\right)}{1-z} n\binom{n-1}{j} z^{n-(j+1)}(1-z)^{j} d z\right) .
\end{aligned}
$$

It is then easy to conclude that $f^{B}(z, n-j, j+1)=n\binom{n-1}{j} z^{n-(j+1)}(1-z)^{j}$ represents the pdf of the beta distribution with parameters $(n-j, j+1)$. To see this, notice that the corresponding pdf is given by $f^{B}(z, n-j, j+1)=\frac{z^{n-j-1}(1-z)^{j} \Gamma(n+1)}{\Gamma(n-j) \Gamma(j+1)}$, where Γ denotes the gamma function. Using $\Gamma(k)=(k-1)!$, this can be written as

$$
f^{B}(z, n-j, j+1)=\frac{z^{n-j-1}(1-z)^{j} n!}{(n-j-1)!(j)!}=n\binom{n-1}{j} z^{n-(j+1)}(1-z)^{j},
$$

which corresponds to the above expression.

Defining $\hat{h}(e, x):=\operatorname{MRTS}\left(e, F^{-1}(x)\right) \frac{f\left(F^{-1}(x)\right)}{1-x}$ and taking into account that the beta distribution characterizes the order statistics from a sample of independent and uniformly distributed random variables, we can rewrite $\frac{B_j(e)}{j}$ as

$$
\frac{B_{j}(e)}{j}=\frac{1}{n} \mathbb{E}\left(\hat{h}(e, Z(n-j: n))\right),
$$

where $Z(n-j:n)$ is the $(n-j)$-th order statistic among n i.i.d. draws from the uniform distribution on $[0,1]$.

Since the order statistics $Z(n-j:n)$ are FOSD-decreasing in j, it follows that $\frac{B_j(e)}{j}$ is maximized at $j=1$ if $\hat{h}(e,x)$ is increasing in x. If this is the case for all e, the winner-takes-all contest is optimal (according to Lemma 2). Likewise, $\frac{B_j(e)}{j}$ is maximized at $j=n-1$ if $\hat{h}(e,x)$ is decreasing in x. If this is the case for all e, the loser-gets-nothing contest is optimal.

$\square$

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the authors used Refine (Refine Technologies) and Claude (Anthropic) as a consistency check on the results and proofs, to flag possible errors and gaps for the authors' attention. All proofs were established and checked independently by the authors, who take full responsibility for the content of the published article.

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