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Yuli Liang*, Chengcheng Hao and Deliang Dai

Abstract Under a model having a Kronecker product covariance structure with compound symmetry or circular symmetry, two-sample hypothesis testing for the equality of two correlation parameters is considered. Different tests are proposed by using the ratio of independent F distributions. Several tests are compared with the proposed ones and practical recommendations are made based on their type I error probabilities and powers. Finally, all mentioned tests are applied to a real data example.

Key words: Kronecker covariance structure, Higher order asymptotics, Ratio of F distributions.

1 Introduction

The study of the patterned covariance matrices with Kronecker product of two components, that is, $\Sigma \otimes \Psi$, where $\Sigma(q \times q)$ and $\Psi(p \times p)$, has raised much attention in recent years. Among others, this structure can be particularly useful to model spatial-temporal dependency simultaneously, where Σ is connected to temporal dependency and Ψ models the dependency over space (see Srivastava *et al.*, 2009, for example). In the Kronecker product of two matrices there are only $\frac{1}{2}(p(p+1) + q(q+1))$ parameters to be estimated as compared to the general case where we need to estimate $pq(pq+1)/2$ parameters. Hence it is also preferable in many analysis when there is not enough data otherwise to estimate the model parameters (Dutilleul, 2018). From an inferential point of view, the Kronecker structure makes the estimation more complicated since the identification problem should be resolved and some restrictions have to be imposed on the parameter space. Then it results in non-explicit maximum likelihood estimates (MLEs) which depend on the choice of restrictions imposed on the covariance matrix (Srivastava *et al.*, 2008).

One interesting extension is when there can be some patterns imposed on matrices Σ and Ψ , the compound symmetry (CS) structure. Roy and Leiva (2008) provide an algorithm to compute

Yuli Liang, Deliang Dai

Department of Economics and Statistics, Linnaeus University, SE-352 52 Växjö, Sweden, e-mail: yuli.liang@lnu.se, deliang.dai@lnu.se

Chengcheng Hao

Department of Data Science, School of Statistics and Information, Shanghai University of International Business and Economics, 201620 Shanghai, China, e-mail: chengcheng.hao@suibe.edu.cn

* Corresponding author

the MLEs of the parameters when both Ψ and Σ have a CS structure or an autoregressive of order one structure. Worsley *et al.* (1991) published real data where glucose measurements were taken at 15 different regions of their brain paired in both hemispheres ($q = 2$). Two groups of 10 subjects ($n_1 = n_2 = 10$) were analyzed by two different experimental techniques. It was of interest to test whether three specific pairs of the regions ($p = 3$) with equal distance for two different groups are equally correlated. This requires the task of the equality test of two correlation parameters and this application will be used as an illustrative example later in this article.

In this paper, we also consider another case where one Kronecker component is CS structure and the one is circular Toeplitz (CT) structure. Circular dependence is common in many applications, especially in signal processing (Olkin and Press, 1969), medical (Louden and Roy, 2010) and educational studies (Steinmetz *et al.*, 2012). A CT matrix means that the correlation between two measurements depends only on the circular distance between them, in other words, on the number of positions between them on a circle.

Testing Kronecker product covariance matrix is often challenging especially under normality assumptions the existence and uniqueness of the MLE should be considered. Hao *et al.* (2016) was one of only a few studies (Lu and Zimmerman, 2005; Srivastava *et al.*, 2008) which have proposed some testing procedures and among others it has been noted that two independent exact F tests exist for testing hypothesis concerning the correlations. Moreover, the methodologies for combining the two F tests have been explored.

Motivated by the above arguments, the aim of this paper is to study two-sample intraclass correlation coefficient tests under a model having a Kronecker product covariance structure with compound symmetry or circular symmetry. We shall use one of the higher order asymptotic procedures that modifies the likelihood ratio test so as to achieve accurate performance in small samples; see Brazzale and Davison (2008). By using those independent minimum sufficient statistics, it will also be noted that different independent tests exist by using the ratio of F distributions.

The rest of this paper is organized as follows. In Section 2 the model and the hypothesis are presented. Several test procedures are investigated in Section 3 and it will also be noted that different independent exact tests exist for testing the equality of the two correlations. The performance of all the tests will be evaluated in Section 4 based on simulated type I error probabilities and power. Finally, the example mentioned earlier will be used to illustrate presented methods in Section 5. The article is summarized in the Discussion section.

2 Model and hypothesis

Let $\mathbf{X}_{k1}, \mathbf{X}_{k2}, \dots, \mathbf{X}_{kn_k}$ is a random sample from the distribution of $\mathbf{X}_k$, so that each $\mathbf{X}_{ki}$ is a $p \times q$ matrix. We assume a matrix normal model having the following structure:

$$\mathbf{X}_{ki} \sim N_{p,q}(\boldsymbol{\mu}_k, \Psi_k, \Sigma_k), \quad i = 1, \dots, n_k, \quad k = 1, 2, \quad (1)$$

where $\boldsymbol{\mu}_k : p \times q$ is an unstructured mean, $\Psi_k : p \times p$ describes the covariance between the rows of $\mathbf{X}_{ki}$ at any given column, and $\Sigma_k : q \times q$ is the corresponding covariance matrix between the columns at any given row. The matrices $\boldsymbol{\mu}_k$, Σ_k and Ψ_k are unknown. Let $\mathbf{I}_p$ be the $p \times p$ identity matrix, $\mathbf{1}_p$ be the $p \times 1$ vector with all elements equal to 1 and $\mathbf{J}_p = \mathbf{1}_p \mathbf{1}_p'$, $\mathbf{P}_{1_p} = \frac{1}{p} \mathbf{J}_p$, $\mathbf{Q}_{1_p} = \mathbf{I}_p - \frac{1}{p} \mathbf{J}_p$, $\mathbf{X}_{kic} = \mathbf{X}_{ki} - \bar{\mathbf{X}}_k$, where $\bar{\mathbf{X}}_k$ are the means of the $\mathbf{X}_{ki}$, $k = 1, 2$. The CS structured matrix Ψ_k can be expressed as

$$\Psi_k = \psi_k [(1 - r_k)\mathbf{I}_p + r_k\mathbf{J}_p], \quad (2)$$

where ψ_k and r_k are unknown parameters. Furthermore, the following two specific structures of Σ_k are of interest:

- (i) Σ_k is a CS matrix, i.e., $\Sigma_k = \sigma_k [(1 - \rho_k)\mathbf{I}_q + \rho_k\mathbf{J}_q]$, where σ_k and ρ_k are unknown parameters.
- (ii) $\Sigma_k = (\sigma_{hl})$ is a CT matrix which depends on $r = [q/2] + 1$ parameters, and for $h, l = 1, \dots, q$,

$$\sigma_{hl} = \begin{cases} \tau_{|l-h|+1}, & \text{if } |l-h| \leq r-1, \\ \tau_{q-|l-h|+1}, & \text{otherwise,} \end{cases}$$

where the τ_j 's are unknown parameters for $j = 1, \dots, r$. For the purpose of identifiability, we shall assume that $\psi_k = 1$ in (2). We shall consider the following hypothesis testing problem concerning the correlation coefficients:

$$H_0 : r_1 = r_2 = r \text{ versus } H_a : -\frac{1}{p-1} < r_1 \neq r_2 < 1. \quad (3)$$

3 Test procedures

In this section we shall consider several possible tests for the hypothesis in (3). First we give the eigenvalues and eigenvectors of a CS and CT matrix respectively, since they are important for the subsequent inference. Let Ψ_k^* denote the matrix Ψ_k under the restriction $\Psi_k = \mathbf{I}$. The distinct eigenvalues of Σ_k and Ψ_k^* in (i) are

$$\begin{cases} \lambda_{k1} = \sigma_k(1 + (q-1)\rho_k) \\ \lambda_{k2} = \sigma_k(1 - \rho_k) \end{cases} \quad \text{and} \quad \begin{cases} \lambda_{k3} = 1 + (p-1)r_k \\ \lambda_{k4} = 1 - r_k \end{cases}. \quad (4)$$

The distinct eigenvalues of the CT structured Σ_k in (ii), i.e., ξ_{kj} , $j = 1, \dots, [q/2] + 1$, are given in the following lemma. The spectral properties of CT matrices can, for example, be found in Basilevsky (1983); Nahtman and von Rosen (2008); Liang *et al.* (2021). It is worth noting that a CS matrix is a special case of a CT matrix so the following lemma can also be applied to a CS matrix.

Lemma 1. *Let $\Sigma : q \times q$ be a CT matrix and let $\lambda_h, h = 1, \dots, q$, be an eigenvalue of Σ .*

(i) *If q is odd, then*

$$\lambda_h = \tau_0 + 2 \sum_{j=1}^{[q/2]} \tau_j \cos(2\pi h j / q).$$

It follows that, $\lambda_h = \lambda_{q-h}$, $h = 1, \dots, q-1$, and there is only one eigenvalue, λ_q , which has multiplicity 1, all other eigenvalues are of multiplicity 2.

If q is even, then

$$\lambda_h = \tau_0 + 2 \sum_{j=1}^{q/2-1} \tau_j \cos(2\pi h j / q) + \tau_{q/2} \cos(\pi h).$$

It follows that, for $h \neq q, q/2 : \lambda_h = \lambda_{q-h}$, there are only two eigenvalues, λ_q and $\lambda_{q/2}$, which have multiplicity 1, and all other eigenvalues are of multiplicity 2.

(ii) The number of distinct eigenvalues for CT matrix is $\lceil \frac{q}{2} \rceil + 1$.

(iii) A set of eigenvectors $(\mathbf{v}_1, \dots, \mathbf{v}_q)$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_q$, is defined by

$$v_{hi} = \frac{1}{\sqrt{q}} (\cos(2\pi ih/q) + \sin(2\pi ih/q)), \quad i, h = 1, \dots, q. \quad (5)$$

The eigenvalues and eigenvectors of a CT matrix in Lemma 1 will be utilized to obtain the minimum sufficient statistics in the next subsection, which in turn can be used to constitute the likelihood function in canonical form (see Section 3.2).

3.1 Minimum sufficient statistics

Let $\boldsymbol{\eta}$ a vector containing all unknown parameters in the covariance matrices $(\boldsymbol{\Psi}_1, \boldsymbol{\Sigma}_1)$ and $(\boldsymbol{\Psi}_2, \boldsymbol{\Sigma}_2)$. Since the hypothesis of interest concerns two components of $\boldsymbol{\eta}$, the mean parameters $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are the nuisance parameters. Therefore we only consider the restricted log-likelihood function of $\boldsymbol{\eta}$, say $\ell_c(\boldsymbol{\eta})$. With the CS structured $\boldsymbol{\Psi}_k^*$ and $\boldsymbol{\Sigma}_k$ in model (1), the minimum sufficient statistics and the corresponding distributions are given in the following theorem by Hao *et al.* (2016).

Theorem 1. When $\boldsymbol{\Sigma}_k$ is assumed to be a CS matrix in model (1), $k = 1, 2$, the minimum sufficient statistics of $\boldsymbol{\eta}$ are $\mathbf{t}_k = (t_{k1}, \dots, t_{k4})'$, where

$$\begin{aligned} t_{k1} &= \sum_{i=1}^{n_k} \text{tr} \{ \mathbf{P}_{1_q} \mathbf{X}_{kic}' \mathbf{P}_{1_p} \mathbf{X}_{kic} \} \sim (-2\theta_{k1})^{-1} \chi^2(n_k - 1), \\ t_{k2} &= \sum_{i=1}^{n_k} \text{tr} \{ \mathbf{P}_{1_q} \mathbf{X}_{kic}' \mathbf{Q}_{1_p} \mathbf{X}_{kic} \} \sim (-2\theta_{k2})^{-1} \chi^2((n_k - 1)(p - 1)), \\ t_{k3} &= \sum_{i=1}^{n_k} \text{tr} \{ \mathbf{Q}_{1_q} \mathbf{X}_{kic}' \mathbf{P}_{1_p} \mathbf{X}_{kic} \} \sim (-2\theta_{k3})^{-1} \chi^2((n_k - 1)(q - 1)), \\ t_{k4} &= \sum_{i=1}^{n_k} \text{tr} \{ \mathbf{Q}_{1_q} \mathbf{X}_{kic}' \mathbf{Q}_{1_p} \mathbf{X}_{kic} \} \sim (-2\theta_{k4})^{-1} \chi^2((n_k - 1)(p - 1)(q - 1)), \end{aligned} \quad (6)$$

and

$$\boldsymbol{\theta}_k(\boldsymbol{\eta}) = (\theta_{k1}, \dots, \theta_{k4})' = -\frac{1}{2} ((\lambda_{k1}\lambda_{k3})^{-1}, (\lambda_{k1}\lambda_{k4})^{-1}, (\lambda_{k2}\lambda_{k3})^{-1}, (\lambda_{k2}\lambda_{k4})^{-1})'. \quad (7)$$

Furthermore, $t_{k1}, \dots, t_{k4}$ are independent.

Next, we present the minimum sufficient statistics and the corresponding distributions when $\boldsymbol{\Sigma}_k$ is a CT matrix in model (1).

Theorem 2. When $\boldsymbol{\Sigma}_k$ is assumed to be a CT matrix in model (1), $k = 1, 2$, the minimum sufficient statistics of $\boldsymbol{\eta}$ are given by

$$t_{k1j} = \sum_{i=1}^{n_k} \{ \mathbf{E}_j \mathbf{X}'_{ic} \mathbf{P}_{1_p} \mathbf{X}_{ic} \} \sim \lambda_{k1} \xi_{kj} \chi^2((n_k - 1)m_j),$$

$$t_{k2j} = \sum_{i=1}^{n_k} \{ \mathbf{E}_j \mathbf{X}'_{ic} \mathbf{Q}_{1_p} \mathbf{X}_{ic} \} \sim \lambda_{k2} \xi_{kj} \chi^2((n_k - 1)(p - 1)m_j),$$

where m_j is the multiplicity of ξ_{kj} , for $j = 1, \dots, [q/2] + 1$.

Proof. Based on the distribution

$$\sum_{i=1}^{n_k} \text{vec } \mathbf{X}_{kic} \text{vec}' \mathbf{X}_{kic} \sim W_{pq}(n_k - 1, \boldsymbol{\Sigma}_k \otimes \boldsymbol{\Psi}_k^*),$$

the restricted log-likelihood function for $\boldsymbol{\mu}_k$ is obtained as follows:

$$\ell_c(\boldsymbol{\eta}) = -\frac{n_k - 1}{2} \ln |\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}^*| - \frac{1}{2} \text{tr} \left\{ \left(\sum_{i=1}^{n_k} \text{vec } \mathbf{X}_{ic} \text{vec}' \mathbf{X}_{ic} \right) (\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Psi}^{*-1}) \right\}. \quad (8)$$

By using the spectral decomposition $\boldsymbol{\Sigma}_k = \sum_{j=1}^r \xi_{kj} \mathbf{E}_j$, and $\boldsymbol{\Psi}_k^* = \lambda_{k3} \mathbf{P}_{1_p} + \lambda_{k4} \mathbf{Q}_{1_p}$, where ξ_{kj} , $j = 1, \dots, r$ are the eigenvalues given by Lemma 1 and λ_{k3} and λ_{k4} are given in equation (4), $\mathbf{E}_j$ is the orthogonal projection onto the corresponding eigenspace. equation (8) can be simplified as

$$\begin{aligned} \ell_c(\boldsymbol{\eta}) = & -\frac{n_k - 1}{2} \ln(\xi_{k1} \lambda_{k3}) - \frac{n_k - 1}{2} \sum_{j=2}^r m_j \ln(\xi_{kj} \lambda_{k3}) \\ & - \frac{(n_k - 1)(p - 1)}{2} \ln(\xi_{k1} \lambda_{k4}) - \frac{(n_k - 1)(p - 1)}{2} \sum_{j=2}^r m_j \ln(\xi_{kj} \lambda_{k4}) \\ & - \frac{1}{2} \xi_{k1}^{-1} \lambda_{k3}^{-1} \sum_{i=1}^{n_k} \text{tr}(\mathbf{P}_{1_q} \mathbf{X}'_{ic} \mathbf{P}_{1_p} \mathbf{X}_{ic}) - \frac{1}{2} \lambda_{k3}^{-1} \sum_{j=2}^r \xi_{kj}^{-1} \left\{ \sum_{i=1}^{n_k} \text{tr}(\mathbf{E}_j \mathbf{X}'_{ic} \mathbf{P}_{1_p} \mathbf{X}_{ic}) \right\} \\ & - \frac{1}{2} \xi_{k1}^{-1} \lambda_{k4}^{-1} \sum_{i=1}^{n_k} \text{tr}(\mathbf{P}_{1_q} \mathbf{X}'_{ic} \mathbf{Q}_{1_p} \mathbf{X}_{ic}) - \frac{1}{2} \lambda_{k4}^{-1} \sum_{j=2}^r \xi_{kj}^{-1} \left\{ \sum_{i=1}^{n_k} \text{tr}(\mathbf{E}_j \mathbf{X}'_{ic} \mathbf{Q}_{1_p} \mathbf{X}_{ic}) \right\}. \end{aligned} \quad (9)$$

The log-likelihood (9) belongs to a curved exponential family, and the minimal sufficient statistic for the unknown covariance parameters is $\mathbf{t} = (t_{k11}, \dots, t_{k1r}, t_{k21}, \dots, t_{k2r})'$, where

$$t_{k1j} = \sum_{i=1}^{n_k} \{ \mathbf{E}_j \mathbf{X}'_{ic} \mathbf{P}_{1_p} \mathbf{X}_{ic} \}, \quad t_{k2j} = \sum_{i=1}^{n_k} \{ \mathbf{E}_j \mathbf{X}'_{ic} \mathbf{Q}_{1_p} \mathbf{X}_{ic} \}, \quad (10)$$

where m_j is the multiplicity of ξ_{kj} , for $j = 1, \dots, r$. It is readily seen that t_{k1j} and t_{k2j} are independent and

$$t_{k1j} \sim \lambda_{k3} \xi_{kj} \chi^2((n_k - 1)m_j), \quad t_{k2j} \sim \lambda_{k4} \xi_{kj} \chi^2((n_k - 1)(p - 1)m_j).$$

□

3.2 Likelihood ratio test

With the CS structured Σ_k , the restricted log-likelihood function $\ell_c(\boldsymbol{\eta})$ can be written in canonical form as

$$\ell_c(\boldsymbol{\eta}) = [(n_1 - 1) c(\boldsymbol{\theta}_1(\boldsymbol{\eta})) + \mathbf{t}'_1 \boldsymbol{\theta}_1(\boldsymbol{\eta})] \times [(n_2 - 1) c(\boldsymbol{\theta}_2(\boldsymbol{\eta})) + \mathbf{t}'_2 \boldsymbol{\theta}_2(\boldsymbol{\eta})],$$

where $\boldsymbol{\theta}_k(\boldsymbol{\eta})$ is given in (7) and is the so called canonical parameterization of $\boldsymbol{\eta}$, and

$$c(\boldsymbol{\theta}_k(\boldsymbol{\eta})) = -\frac{1}{2} [\ln(-\theta_{k1}) + (p-1) \ln(-\theta_{k2}) + (q-1) \ln(-\theta_{k3}) + (p-1)(q-1) \ln(-\theta_{k4})].$$

With the CT structured Σ_k , the restricted log-likelihood function $\ell_c(\boldsymbol{\eta})$ is given in (9). The REML estimators of ξ_{kj} , λ_{k3} and λ_{k4} are given by

$$\begin{aligned} \hat{\xi}_{kj} &= \frac{1}{p(n_k - 1)m_j} \left(\frac{t_{k1j}}{\lambda_{k3}} + \frac{t_{k2j}}{\lambda_{k4}} \right), \\ \hat{\lambda}_{k3} &= \frac{1}{q(n_k - 1)} \sum_{j=1}^r \frac{t_{k1j}}{\xi_{kj}}, \\ \hat{\lambda}_{k4} &= \frac{1}{q(p-1)(n_k - 1)} \sum_{j=1}^r \frac{t_{k2j}}{\xi_{kj}}, \end{aligned}$$

where the statistics t_{k1j} , t_{k2j} and m_j are given in Theorem 2.

Now let $\hat{r}_1$ and $\hat{r}_2$ be the restricted maximum likelihood estimator (REML) of r_1 and r_2 . Based on the restrictive likelihood function, the classical signed log-likelihood ratio statistic can be written as

$$LD_c(r) = \text{sign}(\hat{r}_1 - \hat{r}_2) \left[2 \left\{ \sup_{H_0} \ell_c(\boldsymbol{\eta}) - \sup_{H_1} \ell_c(\boldsymbol{\eta}) \right\} \right]^{1/2}, \quad (11)$$

which sup is the supremum function and $\text{sign}(x)$ is +1 or -1 depending on whether $x > 0$ or $x < 0$, respectively.

It is known that the test statistic given in (11) is asymptotically standard normal distributed under regularity conditions, with the tail area approximations having error $O(n^{-1/2})$. One can show the log-likelihood belongs to a curved exponential family. The MLEs of $\boldsymbol{\eta}$ do not have closed forms under both H_0 and H_1 and are obtained iteratively.

3.3 Higher order procedure

We shall now consider one higher order modification of $LD_c(r)$ in order to have better type I error performance in small samples. The modification that we shall consider is due to DiCiccio *et al.* (2001). The modified statistic, say $LD_{c_{mv}}(r)$, is given by

$$LD_{c_{mv}}(r) = \frac{LD_c(r) - m(\hat{\boldsymbol{\eta}}_{c0})}{\sqrt{1 + v(\hat{\boldsymbol{\eta}}_{c0})}}, \quad (12)$$

where $m(\hat{\boldsymbol{\eta}}_{c0})$ and $1 + v(\hat{\boldsymbol{\eta}}_{c0})$ are the mean and variance of $LD_c(r)$, respectively, which can be estimated from Monte Carlo simulation of $\mathbf{t}_k$ at the parameter value $\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}_{c0}$. Subject to regularity conditions, $LD_{c_{mv}}(r)$ has an asymptotic standard normal distribution, and tail area approximation based on $LD_c(r)$ has relative error $O(n^{-3/2})$.

3.4 Tests based on ratio of independent F distributions

We note that the eigenvalues λ_{k1} and λ_{k2} in (4) are free of r_k , and $\lambda_{k3}/\lambda_{k4} = (1 - r_k)/(1 + (p - 1)r_k)$. Then we have the following distributions:

$$\begin{aligned} c_k t_{k1}/t_{k2} &\sim F_{n_k-1, (n_k-1)(p-1)}, \\ c_k t_{k3}/t_{k4} &\sim F_{(n_k-1)(q-1), (n_k-1)(p-1)(q-1)}, \end{aligned}$$

where $c_k = (p-1) \frac{1-r_k}{1+(p-1)r_k}$. Using this observation, two exact tests can be constructed for testing the null hypothesis in (3), based on the distributions in (6). Construct the following independent test statistics:

$$\begin{aligned} F_1^{CS}(r) &= c_1 \left(\frac{t_{11}}{t_{12}} \right) / c_2 \left(\frac{t_{21}}{t_{22}} \right), \\ F_2^{CS}(r) &= c_1 \left(\frac{t_{13}}{t_{14}} \right) / c_2 \left(\frac{t_{23}}{t_{24}} \right). \end{aligned} \tag{13}$$

Under the null hypothesis $H_0 : r_1 = r_2 = r$, we have the distributions $F_1^{CS}(r) \sim \frac{F_{n_1-1, (n_1-1)(p-1)}}{F_{n_2-1, (n_2-1)(p-1)}}$ and $F_2^{CS}(r) \sim \frac{F_{(n_1-1)(q-1), (n_1-1)(p-1)(q-1)}}{F_{(n_2-1)(q-1), (n_2-1)(p-1)(q-1)}}$. Under the alternative hypothesis $H_a : -\frac{1}{p-1} < r_1 \neq r_2 < 1$, we have $F_1^{CS}(r) \sim \frac{c_1}{c_2} \frac{F_{n_1-1, (n_1-1)(p-1)}}{F_{n_2-1, (n_2-1)(p-1)}}$ and $F_2^{CS}(r) \sim \frac{c_1}{c_2} \frac{F_{(n_1-1)(q-1), (n_1-1)(p-1)(q-1)}}{F_{(n_2-1)(q-1), (n_2-1)(p-1)(q-1)}}$.

When $\boldsymbol{\Sigma}_k$ is a CT matrix, the eigenvalues ξ_{kj} are functions of r_k given in (4). Thus,

$$t_{k1j}/t_{k2j} \sim (\lambda_{k3}/\lambda_{k4}) F_{(n_k-1)m_j, (n_k-1)(p-1)m_j}.$$

Thus $r = [q/2] + 1$ exact tests can be constructed for testing hypothesis in (3):

$$F_j^{CT}(r) = c_1 \left(\frac{t_{11j}}{t_{12j}} \right) / c_2 \left(\frac{t_{21j}}{t_{22j}} \right), \tag{14}$$

where t_{k1j} and t_{k2j} are given in Theorem 2, $k = 1, 2$, $j = 1, \dots, r$. In the next example the proposed tests in (14) will be illustrated when $q = 4$.

Example 1. When $\boldsymbol{\Sigma}_k$ has a 4×4 CT structure, we have the following distributions:

$$\begin{aligned} c_k t_{k11}/t_{k21} &\sim F_{n_k-1, (n_k-1)(p-1)}, \\ c_k t_{k12}/t_{k22} &\sim F_{2(n_k-1), 2(n_k-1)(p-1)}, \\ c_k t_{k13}/t_{k23} &\sim F_{(n_k-1), (n_k-1)(p-1)}, \end{aligned}$$

where $c_k = (p-1) \frac{1-r_k}{1+(p-1)r_k}$. Then we can construct 3 exact tests for testing the hypothesis in (3):

$$\begin{aligned}
F_1^{CT}(r) &= c_1 \left(\frac{t_{111}}{t_{121}} \right) / c_2 \left(\frac{t_{211}}{t_{221}} \right), \\
F_2^{CT}(r) &= c_1 \left(\frac{t_{112}}{t_{122}} \right) / c_2 \left(\frac{t_{212}}{t_{222}} \right), \\
F_3^{CT}(r) &= c_1 \left(\frac{t_{113}}{t_{123}} \right) / c_2 \left(\frac{t_{213}}{t_{223}} \right).
\end{aligned} \tag{15}$$

Under the null hypothesis $H_0 : r_1 = r_2 = r$, we have the distributions:

$$\begin{aligned}
F_1^{CT}(r) &\sim \frac{F_{n_1-1, (n_1-1)(p-1)}}{F_{n_2-1, (n_2-1)(p-1)}}, \\
F_2^{CT}(r) &\sim \frac{F_{2(n_1-1), 2(n_1-1)(p-1)}}{F_{2(n_2-1), 2(n_2-1)(p-1)}}, \\
F_3^{CT}(r) &\sim \frac{F_{(n_1-1), (n_1-1)(p-1)}}{F_{(n_2-1), (n_2-1)(p-1)}}.
\end{aligned}$$

Under the alternative hypothesis $H_a : -\frac{1}{p-1} < r_1 \neq r_2 < 1$, the distributions are given by $F_1^{CT}(r) \sim \frac{c_1}{c_2} \frac{F_{n_1-1, (n_1-1)(p-1)}}{F_{n_2-1, (n_2-1)(p-1)}}$, $F_2^{CT}(r) \sim \frac{c_1}{c_2} \frac{F_{2(n_1-1), 2(n_1-1)(p-1)}}{F_{2(n_2-1), 2(n_2-1)(p-1)}}$ and $F_3^{CT}(r) \sim \frac{c_1}{c_2} \frac{F_{(n_1-1), (n_1-1)(p-1)}}{F_{(n_2-1), (n_2-1)(p-1)}}$.

The exact distributions of the test statistics in (13) are non-standard distributions and the determinations are relatively complex. The exact distributions can be specified by their characteristic functions and therefore can be computed with the use of R package *CharFunToolR* (Gajdoš, 2018) based on numerical inversion of the characteristic functions.

4 Simulation studies

To investigate the performance of the proposed tests in Section 3, we now evaluate their performance in terms of type I error probabilities and power. The results of testing H_0 are in the context of $r_1 = r_2 = 0.9$ and based on 10 000 simulations and we put $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 30$ and $\rho_1 = \rho_2 = 0.5$. In addition, both balanced and unbalanced cases are included in the simulations. For a 5% significance level, Table 1 gives the estimated type I error probabilities of the tests based on the likelihood ratio test (LRT) given in (11), the mean-variance modified statistics (m-v) given in (12), and the two independent test statistics $F_1^{CS}(r)$ and $F_2^{CS}(r)$ given in (13), respectively.

From Tables 1-2, we draw the following conclusions. It is clear that m-v, $F_1^{CS}(r)$, $F_2^{CS}(r)$ and $F_j^{CT}(r)$ ($j = 1, \dots, 3$) are very accurate under all settings and they provide better control of the type I probability, compared to LRT. For the unbalanced cases, $F_1^{CS}(r)$, $F_2^{CS}(r)$ and $F_j^{CT}(r)$ ($j = 1, \dots, 3$) are superior to the other two testing procedures in terms of accuracy while both LRT and m-v produce relatively higher type I error probabilities than the nominal significance level.

We now compare the power performance of the different tests and the results are given in Table 3. The alternatives used are $(r_1, r_2) = (0.1, 0.2)$ and $(r_1, r_2) = (0.7, 0.8)$. Seen from Table 3-4, for the balanced cases, LRT, m-v, $F_2^{CS}(r)$ and $F_2^{CT}(r)$ are all competitive except when p , q and n are all small. In Table 3 all the tests have significantly higher powers when $(r_1, r_2) = (0.7, 0.8)$ than $(r_1, r_2) = (0.1, 0.2)$. For the unbalanced cases, except $F_1^{CS}(r)$ all the tests are competitively powerful when both p and q are relatively large and however the same conclusion cannot be made

when both p and q are small. The overall conclusions based on the numerical results in Tables 1-2 and 3-4 are that the tests based on m-v, $F_2^{CS}(r)$ and $F_2^{CT}(r)$ can be recommended for practical use when we have balanced data on hand. For the unbalanced data we can choose LRT, m-v, $F_2^{CS}(r)$ and $F_2^{CT}(r)$. Since the quantities $m(\hat{\eta}_{c0})$ and $v(\hat{\eta}_{c0})$ in (12) are computed numerically, the test procedure m-v is very computer-intensive even through it provides satisfactory performance in terms of Type I error probabilities and power. For both cases both m-v, $F_2^{CS}(r)$ and $F_2^{CT}(r)$ are accurate in terms of type I error performance, and have reasonable power.

Table 1: Type I error probabilities of testing $H_0 : r_1 = r_2 = 0.9$ for a 5% significance level when Σ_k are CS matrices

	(p, q, n)			
	(3, 2, 5)	(3, 2, 100)	(24, 12, 5)	(24, 12, 100)
LRT	0.0607	0.0496	0.0487	0.0440
m-v	0.0484	0.0532	0.0520	0.0503
$F_1^{CS}(r)$	0.0463	0.0501	0.0458	0.0500
$F_2^{CS}(r)$	0.0480	0.0501	0.0500	0.0499
	(p, q, n_1, n_2)			
	(3, 2, 5, 100)		(24, 12, 5, 100)	
LRT	0.0547		0.0520	
m-v	0.0505		0.0500	
$F_1^{CS}(r)$	0.0494		0.0500	
$F_2^{CS}(r)$	0.0488		0.0500	

Table 2: Type I error probabilities of testing $H_0 : r_1 = r_2 = 0.9$ for a 5% significance level when Σ_k are CT matrices

	(p, q, n)			
	(3, 4, 5)	(3, 4, 100)	(24, 5, 5)	(24, 5, 100)
LRT	0.0687	0.0556	0.0564	0.0520
m-v	0.0440	0.0450	0.0470	0.0501
$F_1^{CT}(r)$	0.0491	0.0500	0.0493	0.0500
$F_2^{CT}(r)$	0.0495	0.0500	0.0499	0.0500
$F_3^{CT}(r)$	0.0487	0.0501	0.0498	0.0500
	(p, q, n_1, n_2)			
	(3, 4, 5, 100)		(24, 5, 5, 100)	
LRT	0.0507		0.0468	
m-v	0.0535		0.0582	
$F_1^{CT}(r)$	0.0495		0.0500	
$F_2^{CT}(r)$	0.0500		0.0500	
$F_3^{CT}(r)$	0.0497		0.0500	

5 Applications of the results - an example

In this section we use the the positron emission tomography data given in Worsley *et al.* (1991) to illustrate our proposed testing methods in Section 3. Researchers are often interested in several selected regions of the brain with closed locations, which tend to be functionally related, such as the Anterior temporal (AT), Midtemporal (MT) and Posttemporal (PT) regions. The locations

Table 3: Powers of the testings $H_0 : r_1 = r_2$ at $\alpha = 0.05$ when Σ_k are CS matrices

		(p, q, n)			
(r_1, r_2)		(3, 2, 5)	(3, 2, 100)	(24, 12, 5)	(24, 12, 100)
(0.1, 0.2)	LRT	0.0708	0.3386	0.5756	1.0000
	m-v	0.1007	0.2792	0.6150	1.0000
	$F_1^{CS}(r)$	0.0701	0.2912	0.1344	0.9303
	$F_2^{CS}(r)$	0.0739	0.3067	0.6644	1.0000
(0.7, 0.8)	LRT	0.1376	0.9943	0.7903	1.0000
	m-v	0.1138	0.9840	0.7902	1.0000
	$F_1^{CS}(r)$	0.0890	0.6182	0.1152	0.8253
	$F_2^{CS}(r)$	0.0984	0.6350	0.5218	1.0000
		(p, q, n_1, n_2)			
(r_1, r_2)		(3, 2, 5, 100)		(24, 12, 5, 100)	
(0.1, 0.2)	LRT	0.0731		0.8482	
	m-v	0.1005		0.8791	
	$F_1^{CS}(r)$	0.0947		0.2822	
	$F_2^{CS}(r)$	0.0888		0.8968	
(0.7, 0.8)	LRT	0.2201		0.9618	
	m-v	0.2419		0.9611	
	$F_1^{CS}(r)$	0.1404		0.2229	
	$F_2^{CS}(r)$	0.1297		0.7835	

Table 4: Powers of the testings $H_0 : r_1 = r_2$ at $\alpha = 0.05$ when Σ_k are CT matrices

		(p, q, n)			
(r_1, r_2)		(3, 4, 5)	(3, 4, 100)	(24, 5, 5)	(24, 5, 100)
(0.1, 0.2)	LRT	0.0228	0.3050	0.1479	1.0000
	m-v	0.0800	0.6167	0.3329	1.0000
	$F_1^{CT}(r)$	0.0736	0.3073	0.1326	0.9364
	$F_2^{CT}(r)$	0.0894	0.4747	0.2132	0.9965
	$F_3^{CT}(r)$	0.0746	0.2917	0.2096	0.9985
(0.7, 0.8)	LRT	0.0634	0.8859	0.1220	0.9992
	m-v	0.1323	0.9761	0.2534	1.0000
	$F_1^{CT}(r)$	0.0958	0.6427	0.1130	0.8333
	$F_2^{CT}(r)$	0.1312	0.8753	0.1701	0.9747
	$F_3^{CT}(r)$	0.0996	0.6302	0.1702	0.9792
		(p, q, n_1, n_2)			
(r_1, r_2)		(3, 4, 5, 100)		(24, 5, 5, 100)	
(0.1, 0.2)	LRT	0.0850		0.4371	
	m-v	0.0866		0.4186	
	$F_1^{CT}(r)$	0.0904		0.2641	
	$F_2^{CT}(r)$	0.1153		0.3963	
	$F_3^{CT}(r)$	0.0918		0.3840	
(0.7, 0.8)	LRT	0.1709		0.3143	
	m-v	0.1752		0.2954	
	$F_1^{CT}(r)$	0.1347		0.2106	
	$F_2^{CT}(r)$	0.1994		0.3130	
	$F_3^{CT}(r)$	0.1391		0.3016	

are defined as the centers of the regions in one hemisphere of the brain, measured in centimeters, taken from the brain atlas. We recall from Section 1 that the sample size here is $n = 10$ for each

group. Furthermore, we also have $p = 3$ and $q = 2$. We are interested in testing whether the three specific pairs (AT, MT and PT) of the regions ($p = 3$) with equal distance for two groups (with two different experimental techniques) are equally correlated.

In this article, different regions, as well as different hemispheres are considered exchangeable (equicorrelated) among themselves. Thus, it is reasonable to assume that both $\text{vec } \mathbf{X}_{1i}$ and $\text{vec } \mathbf{X}_{2i}$, $i, \dots, 10$, in the samples have the covariance matrices of the form $\boldsymbol{\Sigma}_1 \otimes \boldsymbol{\Psi}_1$ and $\boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Psi}_2$, respectively, where $(\boldsymbol{\Psi}_1, \boldsymbol{\Sigma}_1)$ and $(\boldsymbol{\Psi}_2, \boldsymbol{\Sigma}_2)$ have the structures given in (2). Note, when $q = 2$ CT and CS become the same pattern. We shall test the following hypothesis:

$$H_0 : r_1 = r_2 = r \text{ against } H_a : -\frac{1}{p-1} < r_1 \neq r_2 < 1 \text{ given CS structures.}$$

We consider the test statistics LRT, m-v, $F_1^{CS}(r)$ and $F_2^{CS}(r)$. The test statistics and the p-values are given in Table 5. Note, the quantiles (and thus also p-value) of $F_1^{CS}(r)$ and $F_2^{CS}(r)$ exact distributions are computed with the use of R package *CharFunToolR*. The results indicate that the null hypothesis $H_0 : r_1 = r_2$ cannot be rejected with $\alpha = 0.05$.

Table 5: Test statistics and p-values for the positron emission tomography data ($\alpha = 0.05$). Test statistics and p-value are reported for the hypotheses $r_1 = r_2$ against $r_1 \neq r_2$ given CS structures.

	LRT	m-v	$F_1^{CS}(r)$	$F_2^{CS}(r)$
stat	0.3219	0.4037	1.0453	1.7568
p-val	0.7475	0.6864	0.5267	0.7414

6 Discussion

The tests $F_1^{CS}(r)$ and $F_2^{CS}(r)$ as well as $F_j^{CT}(r)$, $j = 1, \dots, r$ proposed in this paper provide a valid alternative to tests for the equality of two correlation parameters in multivariate models with Kronecker product covariance structure with compound symmetric or circular Toeplitz patterns. In some cases these tests are more accurate and powerful than other presented tests although $F_1^{CS}(r)$ has lower power than $F_2^{CS}(r)$. Concerning the test statistics $F_j^{CT}(r)$, $j = 1, \dots, r$, we may incorporate the information of different statistics and combine them. Another advantage of the tests is the explicit null distributions of test statistics and it is easy to implement.

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